

A WEAK-STRONG UNIQUENESS PRINCIPLE FOR THE MULLINS–SEKERKA EQUATION

JULIAN FISCHER, SEBASTIAN HENSEL, TIM LAUX, AND THERESA M. SIMON

ABSTRACT. We establish a weak-strong uniqueness principle for the two-phase Mullins–Sekerka equation in ambient dimension $d = 2$ and 3 : As long as a classical solution to the evolution problem exists, any weak De Giorgi type varifold solution (see for this notion the recent work of Stinson and the second author, Arch. Ration. Mech. Anal. **248**, 8, 2024) must coincide with it. In particular, in the absence of geometric singularities such weak solutions do not introduce a mechanism for (unphysical) non-uniqueness. We also derive a stability estimate with respect to changes in the data. Our method is based on the notion of relative entropies for interface evolution problems, a reduction argument to a perturbative graph setting, and a stability analysis in this perturbative regime relying crucially on the gradient flow structure of the Mullins–Sekerka equation.

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1. INTRODUCTION

1.1. Context. This work addresses the uniqueness and stability properties of the Mullins–Sekerka equation, a free boundary problem modeling volume-preserving phase separation and coarsening processes for various physical situations. The model was first introduced by Mullins and Sekerka in [23] and can be derived from thermodynamic principles [12]. It also arises as the sharp-interface limit of the Cahn–Hilliard equation ([1], [3] and [21]).

The Mullins–Sekerka equation can be viewed as a geometric third-order partial differential equation. This viewpoint is useful for constructing classical solutions ([30] and [5]), studying long-term asymptotics via a center manifold [6], and it also directly predicts the typical scaling behavior of the induced coarsening process. Naturally, as a higher-order evolution equation, the Mullins–Sekerka equation does not admit a comparison principle, which makes uniqueness questions subtle. Additionally, certain symmetric singularities can lead to physical non-uniqueness. However, it is conjectured that before the onset of singularities, the evolution is unique. We prove this uniqueness property in a large class of weak solutions by controlling a suitable relative entropy functional. As our relative entropy controls the distance of the weak and strong solutions, we additionally derive the stability of solutions with respect to perturbations of their initial data.

Our viewpoint is variational, interpreting the Mullins–Sekerka equation as a gradient flow. Albeit a well-appreciated fact in the community and obvious for smooth solutions, the rigorous definition and an existence proof of weak solutions based on the gradient flow structure has only been given recently by Stinson and the second author [15]. In the present work, we show that it is this very structure that guarantees the weak-strong uniqueness of the flow. It is natural that the gradient flow structure of the Mullins–Sekerka equation plays a crucial role in its uniqueness properties as it also lies at the heart of many other properties of the equation, such as the long-term asymptotics ([16] and [24]) or its (formal) recovery as a sharp interface limit of the Cahn–Hilliard equation [21].

Next to [15], there are several other weak solution concepts for the Mullins–Sekerka equation. The pioneering work [22] of Luckhaus and Sturzenhecker constructs weak solutions via an implicit time discretization. They rely on an assumption on the convergence of energies that was later removed by Röger [26], using a result of Schätzle [27]. We point out that our results also apply to the well-known weak solutions constructed by Luckhaus and Sturzenhecker satisfying a sharp energy-dissipation inequality.

1.2. Weak-strong uniqueness principles for interface evolution. In general, there are two theoretical frameworks for proving weak-strong uniqueness for interface evolution equations. Evans and Spruck [7] and independently Chen, Giga and Goto [4] provided a notion of viscosity solution for mean curvature flow. This powerful framework relies on the comparison principle and is therefore only applicable to certain second-order geometric evolution equations.

More recently, the authors have developed a novel framework based on the dissipative nature of many interface evolution problems. The key idea is to monitor a suitable relative entropy functional that controls the distance between a weak and a strong solution. For the two-phase Navier–Stokes equation, this was carried out in [8], and for multiphase mean curvature flow in [9]. Also for other geometric

motions that do not admit a geometric comparison principle, this method has been shown to be applicable, cf. ([18] and [19]), and it even gives new results for classical problems which admit a comparison principle ([14] and [20]).

In the present work, we face several novel difficulties due to the nonlocal structure of the Mullins–Sekerka equation. To overcome them, we first derive a preliminary version of our stability estimate of the relative entropy. When estimating the right-hand side, a key challenge is to define an appropriate chemical potential to compare the dissipation of the weak solution to the one of the strong solution. To this end, we distinguish two cases. If at a given time, the weak solution cannot be parametrized as a graph over the strong solution, we show that the dissipation of the weak solution dominates all other terms. On the other hand, at times when the weak solution can be written as a graph over the strong solution, we are in a perturbative setting. We construct an auxiliary chemical potential which is harmonic in the two phases of the weak solution but attains (an extension of) the mean curvature of the strong solution as its boundary condition on the weak interface. Finally, we use the stability of the Dirichlet-to-Neumann operator to derive a stability estimate for the relative entropy also in the perturbative case.

1.3. Description of the model. Let $d \in \{2, 3\}$, let $T \in (0, \infty)$ be a finite time horizon, let $\Omega \subset \mathbb{R}^d$ be a bounded domain with C^2 -boundary $\partial\Omega$, and let $A := \bigcup_{t \in [0, T)} A(t) \times \{t\}$ be the space-time track of an evolving family of subsets $A(t) \subset \Omega$, all of which are sets of finite perimeter in \mathbb{R}^d such that it holds $\partial A(t) \cap \Omega = \partial^* A(t) \cap \Omega$ for all $t \in [0, T)$. We then consider the Mullins–Sekerka problem

$$\begin{aligned}
(1a) \quad & -\Delta u = 0 && \text{in } \bigcup_{t \in (0, T)} (\Omega \setminus \partial A(t)) \times \{t\}, \\
(1b) \quad & -\llbracket (\mathbf{n}_{\partial A} \cdot \nabla) u \rrbracket \mathbf{n}_{\partial A} = V_{\partial A} && \text{on } \bigcup_{t \in (0, T)} (\partial A(t) \cap \Omega) \times \{t\}, \\
(1c) \quad & u \mathbf{n}_{\partial A} = H_{\partial A} && \text{on } \bigcup_{t \in (0, T)} (\partial A(t) \cap \Omega) \times \{t\}, \\
(1d) \quad & (\mathbf{n}_{\partial \Omega} \cdot \nabla) u = 0 && \text{on } \bigcup_{t \in (0, T)} \partial \Omega \times \{t\}.
\end{aligned}$$

Here, for all $t \in [0, T)$ we denote by $\mathbf{n}_{\partial A}(\cdot, t)$ the unit normal vector field along the interface $\partial A(t) \cap \Omega$ pointing inside $A(t)$, by $H_{\partial A}(\cdot, t)$ the mean curvature vector field along $\partial A(t) \cap \Omega$, by $V_{\partial A}(\cdot, t)$ the normal velocity of $\partial A(t) \cap \Omega$, and by $\mathbf{n}_{\partial \Omega}$ the unit normal vector field along the domain boundary $\partial \Omega$ pointing inside Ω . Furthermore, for all $t \in [0, T)$ the quantity $\llbracket (\mathbf{n}_{\partial A} \cdot \nabla) u \rrbracket(\cdot, t)$ denotes the jump of $(\mathbf{n}_{\partial A} \cdot \nabla) u$ across $\partial A(t) \cap \Omega$, with the orientation chosen such that for all sufficiently regular vector fields $f: \bar{\Omega} \rightarrow \mathbb{R}^d$ and scalar fields $\eta: \bar{\Omega} \rightarrow \mathbb{R}$ it holds (dropping for notational simplicity the time variable)

$$\begin{aligned}
(2) \quad & - \int_{\Omega \setminus \partial A} (\nabla \cdot f) \eta \, dx = \int_{\Omega} f \cdot \nabla \eta \, dx + \int_{\partial A \cap \Omega} \llbracket \mathbf{n}_{\partial A} \cdot f \rrbracket \eta \, d\mathcal{H}^{d-1} \\
& \quad \quad \quad + \int_{\partial \Omega} (\mathbf{n}_{\partial \Omega} \cdot f) \eta \, d\mathcal{H}^{d-1}.
\end{aligned}$$

The associated energy is given by the interface area functional

$$E[A(t)] := \mathcal{H}^{d-1}(\partial A(t) \cap \Omega).$$

Assuming that the evolving interface intersects the domain boundary only orthogonally¹, we may compute at least on a formal level (dropping again for notational simplicity the time variable)

$$\begin{aligned} \frac{d}{dt} E[A] &= - \int_{\partial A \cap \Omega} H_{\partial A} \cdot V_{\partial A} d\mathcal{H}^{d-1} \\ &\stackrel{(1c)}{=} - \int_{\partial A \cap \Omega} u(\mathbf{n}_{\partial A} \cdot V_{\partial A}) d\mathcal{H}^{d-1} \\ &\stackrel{(1b)}{=} \int_{\partial A \cap \Omega} u[(\mathbf{n}_{\partial A} \cdot \nabla u)] d\mathcal{H}^{d-1} \\ &\stackrel{(2),(1a),(1d)}{=} - \int_{\Omega} |\nabla u|^2 dx, \end{aligned}$$

which reveals the formal gradient flow structure of the equation. Next to this energy dissipation relation, we also have formally

$$\begin{aligned} \frac{d}{dt} \int_A \zeta dx &= \int_A \partial_t \zeta dx - \int_{\partial A \cap \Omega} \zeta(\mathbf{n}_{\partial A} \cdot V_{\partial A}) d\mathcal{H}^{d-1} \\ (3) \quad &\stackrel{(1b)}{=} \int_A \partial_t \zeta dx + \int_{\partial A \cap \Omega} \zeta[(\mathbf{n}_{\partial A} \cdot \nabla u)] d\mathcal{H}^{d-1} \\ &\stackrel{(2),(1a),(1d)}{=} \int_A \partial_t \zeta dx - \int_{\Omega} \nabla u \cdot \nabla \zeta dx \end{aligned}$$

for all sufficiently regular test functions $\zeta: \bar{\Omega} \times [0, T] \rightarrow \mathbb{R}$. In particular, it follows that the volume of the evolving phase is preserved under the flow:

$$\int_{A(t)} 1 dx = \int_{A(0)} 1 dx \quad \text{for all } t \in [0, T].$$

Definition 1 below provides a suitable weak formulation of the Mullins–Sekerka problem (1a)–(1d) in the framework of sets of finite perimeter and varifolds. Strong solutions of (1a)–(1d) will be modelled on the basis of a smoothly evolving phase $\mathcal{A} = \bigcup_{t \in [0, T^*)} \mathcal{A}(t) \times \{t\}$ with time horizon $T^* \in (0, T)$, where for each $t \in [0, T^*)$ the open set $\mathcal{A}(t) \subset \Omega$ is assumed to consist of finitely many simply connected components each of which is compactly supported within Ω .

2. MAIN RESULT

Our main result establishes the uniqueness of classical solutions in a large class of weak solutions. As our method is a relative-entropy inequality, we additionally obtain stability for perturbations of the initial data.

Theorem 1 (Weak-strong uniqueness and quantitative stability for the Mullins–Sekerka equation). *Let $d \in \{2, 3\}$. Consider $A = \bigcup_{t \in (0, T)} A(t) \times \{t\}$ and $\mu = (\mu_t)_{t \in (0, T)}$ such that (A, μ) is a De Giorgi type varifold solution for the Mullins–Sekerka problem (1a)–(1d) in the sense of Definition 1 with time horizon $T \in (0, \infty)$ and initial data $A(0)$. Furthermore, let a smoothly evolving phase $\mathcal{A} = \bigcup_{t \in [0, T_*)} \mathcal{A}(t) \times \{t\}$ with time horizon $T_* \in (0, T)$ be given, the motion law of which is subject to*

¹At the level of a strong solution, we will always assume in the present work that the evolving interface is compactly supported within the domain. We will therefore be able to neglect almost all problems in connection with potential contact point dynamics.

Mullins–Sekerka flow (1a)–(1d). We finally assume that for all $t \in [0, T_*)$ the open set $\mathcal{A}(t) \subset \Omega$ is compactly supported within Ω .

There exists an error functional $(0, T_*) \ni t \mapsto E[A, \mu|\mathcal{A}](t) \in L_{loc}^\infty((0, T_*); [0, \infty))$ between the two solutions such that

$$(4) \quad \begin{aligned} E[A, \mu|\mathcal{A}](t) &= 0 \\ \iff \mathcal{L}^d(A(t)\Delta\mathcal{A}(t)) &= 0, \mu_t = \mathcal{H}^{d-1}\llcorner(\partial^*A(t) \cap \Omega) \otimes (\delta_{\mathbf{n}_{\partial^*A}(x,t)})_{x \in (\partial^*A(t) \cap \Omega)}, \end{aligned}$$

and for almost every $T' \in (0, T_*)$ there exists a constant $C(T') > 0$ such that

$$(5) \quad E[A, \mu|\mathcal{A}](T') \leq E[A, \mu|\mathcal{A}](0) + C(T') \int_0^{T'} E[A, \mu|\mathcal{A}](t) dt.$$

In particular,

$$(6) \quad \begin{aligned} E[A, \mu|\mathcal{A}](0) &= 0 \\ \implies \mathcal{L}^d(A(t)\Delta\mathcal{A}(t)) &= 0, \mu_t = \mathcal{H}^{d-1}\llcorner(\partial^*A(t) \cap \Omega) \otimes (\delta_{\mathbf{n}_{\partial^*A}(x,t)})_{x \in (\partial^*A(t) \cap \Omega)} \\ &\text{for a.e. } t \in (0, T_*). \end{aligned}$$

The class of weak solutions in which our weak-strong uniqueness statement holds is described in detail in the following definition. Those solutions were constructed recently by Stinson and the second author [15, Theorem 1 and Lemma 3]. These De Giorgi type varifold solutions are a suitable version of curves of maximal slope for general gradient flows, cf. [2], in the context of the Mullins–Sekerka equation. That means, they characterize the flow by an optimal energy dissipation inequality, see (7f) below. It is worth mentioning that any weak solution in the sense of Luckhaus and Sturzenhecker [22] that satisfies the sharp energy dissipation inequality is such a De Giorgi type varifold solution (with density $\rho = 1$ and equal potentials $u = w$), which means that our main result shows in particular the weak-strong uniqueness in this smaller class as well.

Definition 1 (De Giorgi type varifold solutions of the Mullins–Sekerka equation). Let $d \in \{2, 3\}$, let $T \in (0, \infty)$ be a finite time horizon, let $\Omega \subset \mathbb{R}^d$ be a bounded domain with C^2 -boundary $\partial\Omega$, let $A(0) \subset \Omega$ be an open subset of Ω such that $A(0)$ has finite perimeter in \mathbb{R}^d , and define the associated canonical oriented varifold $\mu_0 := \mathcal{H}^{d-1}\llcorner(\partial^*A(0) \cap \Omega) \otimes (\delta_{\mathbf{n}_{\partial^*A(0)}})_{x \in (\partial^*A(0) \cap \Omega)} \in \mathbf{M}(\overline{\Omega} \times \mathbb{S}^{d-1})$.

Consider a family $(A(t))_{t \in (0, T)}$ of open subsets of Ω , define χ_A as the characteristic function associated with the space-time set $A := \bigcup_{t \in (0, T)} A(t) \times \{t\}$, and consider a family $\mu = (\mu_t)_{t \in (0, T)}$ of oriented varifolds $\mu_t \in \mathbf{M}(\overline{\Omega} \times \mathbb{S}^{d-1})$, $t \in (0, T)$, such that μ_t is $(d-1)$ -integer rectifiable for a.e. $t \in (0, T)$.² We call (A, μ) a *De Giorgi type varifold solution of the Mullins–Sekerka problem (1a)–(1d) with time horizon T and initial data $A(0)$* if (χ_A, μ) is an admissible pair in the precise sense of [15, Definition 3] (with respect to $\alpha = \frac{\pi}{2}$) and

i) (Finite surface area and preserved mass) It holds

$$(7a) \quad \chi_A \in L^\infty(0, T; BV(\mathbb{R}^d; \{0, 1\})), \int_{A(t)} 1 dx = \int_{A(0)} 1 dx \quad \text{for a.e. } t \in (0, T);$$

²An oriented varifold μ is called integer rectifiable if its canonically associated general varifold $\widehat{\mu}$, cf. for an example (202), is integer rectifiable.

- ii) (Kinetic potential) There exists $u \in L^2(0, T; H^1(\Omega))$ with $\int_{\Omega} u(\cdot, t) dx = 0$ for a.e. $t \in (0, T)$ such that

$$(7b) \quad \begin{aligned} & \int_{A(T')} \zeta(\cdot, T') dx - \int_{A(0)} \zeta(\cdot, 0) dx \\ &= \int_0^{T'} \int_{A(t)} \partial_t \zeta dx dt - \int_0^{T'} \int_{\Omega} \nabla u \cdot \nabla \zeta dx dt \end{aligned}$$

for a.e. $T' \in (0, T)$ and all $\zeta \in C_{cpt}^{\infty}(\overline{\Omega} \times [0, T])$;

- iii) (Curvature potential for mean curvature) there exist $w \in L^2(0, T; H^1(\Omega))$ and $C_w = C_w(d, \Omega, T, A(0)) > 0$ such that it holds

$$(7c) \quad \|w(\cdot, t)\|_{H^1(\Omega)} \leq C_w(1 + \|(\nabla w)(\cdot, t)\|_{L^2(\Omega)}),$$

$$(7d) \quad \Delta w(\cdot, t) = 0 \text{ in } \Omega \setminus \overline{\partial^* A(t)}, \quad (\mathbf{n}_{\partial\Omega} \cdot \nabla)w(\cdot, t) = 0 \text{ on } \partial\Omega,$$

for a.e. $t \in (0, T)$, and such that w satisfies the isotropic Gibbs–Thomson law (1c) in the sense that

$$(7e) \quad \int_{\overline{\Omega} \times \mathbb{S}^{d-1}} (\text{Id} - \mathbf{p} \otimes \mathbf{p}) : \nabla B(x) d\mu_t(x, \mathbf{p}) = \int_{A(t)} \nabla \cdot (w(\cdot, t)B) dx$$

for a.e. $t \in (0, T)$ and all $B \in C^1(\overline{\Omega}; \mathbb{R}^d)$ such that $B \cdot \mathbf{n}_{\partial\Omega} = 0$ along $\partial\Omega$;

- iv) (Energy dissipation inequality) and finally the energy dissipation property in the form of

$$(7f) \quad E[\mu_{T'}] + \int_0^{T'} \int_{\Omega} \frac{1}{2} |\nabla u|^2 dx dt + \int_0^{T'} \int_{\Omega} \frac{1}{2} |\nabla w|^2 dx dt \leq E[\mu_0]$$

is satisfied for a.e. $T' \in (0, T)$, where the energy functional is defined by

$$(7g) \quad E[\mu_t] := |\mu_t|_{\mathbb{S}^{d-1}}(\overline{\Omega})$$

for all $t \in [0, T)$, where $|\mu_t|_{\mathbb{S}^{d-1}} \in \mathbf{M}(\overline{\Omega})$ denotes the mass measure of the oriented varifold μ_t .

Remark 2. That μ_t is indeed integer rectifiable for a.e. $t \in (0, T)$ in the precise sense of Footnote 2 — in contrast to the slightly weaker statement of [15, Definition 3, item i)], i.e., that the mass measure of μ_t is $(d-1)$ -integer rectifiable for a.e. $t \in (0, T)$ — is in fact proven in [15, Proof of Theorem 1, Step 6].

3. OVERVIEW OF THE STRATEGY

For the rest of the paper, we consider $d \in \{2, 3\}$ and fix both a De Giorgi type varifold solution (A, μ) with initial data $A(0)$ and time horizon $T \in (0, \infty)$ and a smoothly evolving solution $\mathcal{A} = \bigcup_{t \in [0, T_*)} \mathcal{A}(t) \times \{t\}$ with $T_* \in (0, T)$ for the Mullins–Sekerka problem (1a)–(1d).

We require the following structural assumptions on \mathcal{A} . For every $t \in (0, T_*)$, the phase $\mathcal{A}(t)$ consists of finitely many simply connected components (the number of which is constant in time), all of which are open sets with smooth boundary and being compactly supported within Ω . We also assume that there exists $\ell \in C_{loc}^1([0, T_*]; (0, 1))$ such that for every $t \in (0, T_*)$ the set $B_{\ell(t)}(\partial\mathcal{A}(t)) = \{x \in \mathbb{R}^d : \text{dist}(x, \partial\mathcal{A}(t)) < \ell(t)\}$ is a regular tubular neighborhood for the interface $\partial\mathcal{A}(t) = \partial\mathcal{A}(t) \cap \Omega$ satisfying

$$(8) \quad B_{\ell(t)}(\partial\mathcal{A}(t)) \subset \{x \in \Omega : \text{dist}(x, \partial\Omega) > \ell(t)\}.$$

As usual, the nearest point projection onto $\partial\mathcal{A}(t)$, denoted by $P_{\partial\mathcal{A}}(\cdot, t)$, together with the associated signed distance function, denoted by $s_{\partial\mathcal{A}}(\cdot, t)$, yield a smooth change of variables

$$(9) \quad B_{\ell(t)}(\partial\mathcal{A}(t)) \ni x \mapsto (P_{\partial\mathcal{A}}(\cdot, t), s_{\partial\mathcal{A}}(\cdot, t)) \in \partial\mathcal{A}(t) \times (-\ell(t), \ell(t)),$$

where the signed distance function is oriented according to

$$(10) \quad \nabla s_{\partial\mathcal{A}} = \mathbf{n}_{\partial\mathcal{A}}.$$

Analogously, we assume that $\ell(t)$ is an admissible tubular neighborhood width for $\partial\Omega$ for all $t \in (0, T_*)$.

In order to introduce a suitable error functional between the two solutions A and \mathcal{A} , we will later choose extensions of the strong normal

$$(11) \quad \xi \in W_{loc}^{1,\infty}((0, T_*); W^{1,\infty}(\bar{\Omega}; \mathbb{R}^d)) \cap L_{loc}^\infty((0, T_*); W^{2,\infty}(\bar{\Omega}; \mathbb{R}^d))$$

and the signed distance function, a weight (which one should think of as a truncated version of the signed distance function)

$$(12) \quad \vartheta \in W_{loc}^{1,\infty}((0, T_*); W^{1,\infty}(\bar{\Omega})) \cap L_{loc}^\infty((0, T_*); W^{2,\infty}(\bar{\Omega}))$$

satisfying as a bare minimum

$$(13) \quad \xi = \nabla s_{\partial\mathcal{A}} \text{ in } \bigcup_{t \in [0, T_*)} B_{\frac{\ell(t)}{2}}(\partial\mathcal{A}(t)) \times \{t\}, \quad |\xi| \leq 1 \text{ on } \bar{\Omega} \times [0, T_*),$$

$$(14) \quad \vartheta = -\frac{s_{\partial\mathcal{A}}}{\ell^2} \text{ in } \bigcup_{t \in [0, T_*)} B_{\frac{\ell(t)}{2}}(\partial\mathcal{A}(t)) \times \{t\},$$

$$(15) \quad \vartheta < 0 \text{ in } \bigcup_{t \in [0, T_*)} \mathcal{A}(t) \times \{t\}, \quad \vartheta > 0 \text{ in } \bigcup_{t \in [0, T_*)} (\Omega \setminus \overline{\mathcal{A}(t)}) \times \{t\},$$

$$(16) \quad \xi = 0 \text{ in } \bigcup_{t \in [0, T_*)} (\Omega \setminus B_{\frac{3}{4}\ell(t)}(\partial\mathcal{A}(t))) \times \{t\}.$$

Based on such a pair (ξ, ϑ) , we define for all $t \in [0, T_*)$

$$(17) \quad E_{rel}[A, \mu|\mathcal{A}](t) := E[\mu_t] - \int_{\partial^* A(t) \cap \Omega} \mathbf{n}_{\partial^* A}(\cdot, t) \cdot \xi(\cdot, t) d\mathcal{H}^{d-1},$$

$$(18) \quad E_{vol}[A|\mathcal{A}](t) := \int_{A(t) \Delta \mathcal{A}(t)} |\vartheta(\cdot, t)| dx,$$

$$(19) \quad E[A, \mu|\mathcal{A}](t) := E_{rel}[A, \mu|\mathcal{A}](t) + E_{vol}[A|\mathcal{A}](t).$$

Defining the Radon–Nikodým derivative

$$(20) \quad \varrho_t := \frac{\mathcal{H}^{d-1} \llcorner (\partial^* A(t) \cap \Omega)}{|\mu_t|_{\mathbb{S}^{d-1} \llcorner \Omega}} \in [0, 1],$$

it follows from [15, Definition 3, item ii)] that $\varrho_t = 0$ outside $\partial^* A(t) \cap \Omega$ and

$$(21) \quad \varrho_t \in (2\mathbb{N} - 1)^{-1} \quad \text{on } \partial^* A(t) \cap \Omega,$$

so that we may rewrite the relative entropy (17) in the form of

$$(22) \quad \begin{aligned} E_{rel}[A, \mu|\mathcal{A}](t) &= |\mu_t|_{\mathbb{S}^{d-1}}(\partial\Omega) + \int_{\Omega \cap \{\varrho_t \leq \frac{1}{3}\}} (1 - \rho_t) d|\mu_t|_{\mathbb{S}^{d-1}} \\ &+ \int_{\partial^* A(t) \cap \Omega} (1 - n_{\partial^* A} \cdot \xi)(\cdot, t) d\mathcal{H}^{d-1}. \end{aligned}$$

Furthermore, due to (16) it follows again from [15, Definition 3, item ii)] that

$$(23) \quad E_{rel}[A, \mu|\mathcal{A}](t) = \int_{\overline{\Omega} \times \mathbb{S}^{d-1}} 1 - p \cdot \xi(x, t) d\mu_t(x, p).$$

In order to suitably control the time evolution of the error functional $E[A, \mu|\mathcal{A}]$, a third construction will play a prominent role given by a vector field

$$(24) \quad B \in L_{loc}^\infty((0, T_*); W^{2,\infty}(\overline{\Omega}; \mathbb{R}^d))$$

satisfying at least

$$(25) \quad B = 0 \text{ in } \bigcup_{t \in [0, T_*]} (\Omega \setminus B_{\frac{3}{4}\varrho(t)}(\partial\mathcal{A}(t))) \times \{t\}.$$

One should think of B as an extension of the normal velocity of the smoothly evolving solution \mathcal{A} :

$$(26) \quad B(\cdot, t) = V_{\partial\mathcal{A}}(P_{\partial\mathcal{A}}(\cdot, t), t) \text{ on } B_{\frac{\varrho(t)}{2}}(\partial\mathcal{A}(t)), t \in [0, T_*].$$

With these definitions in place, we will now give an overview on our strategy for the proof of Theorem 1.

3.1. Preliminary relative entropy inequality. The first ingredient is a preliminary estimate for the time evolution of the error functional $E[A, \mu|\mathcal{A}]$ defined in (19), just working under the minimal assumptions (11)–(16) and (24)–(25) for the associated constructions (ξ, ϑ, B) . The goal here is to identify the structure of the terms we will need to estimate by the error functional.

Lemma 3 (Preliminary relative entropy inequality). *For a.e. $T' \in (0, T)$, it holds*

$$(27) \quad \begin{aligned} &E_{rel}[A, \mu|\mathcal{A}](T') \\ &\leq E_{rel}[A, \mu|\mathcal{A}](0) + \int_0^{T'} R_{dissip}(t) + R_{\partial_t \xi}(t) + R_{\nabla B}(t) + R_{varifold/BV}(t) dt \end{aligned}$$

and

$$(28) \quad E_{vol}[A|\mathcal{A}](T') = E_{vol}[A|\mathcal{A}](0) + \int_0^{T'} U_{dissip}(t) + U_{\partial_t \vartheta}(t) + U_{\nabla B}(t) dt,$$

where we defined

$$(29) \quad \begin{aligned} R_{dissip}(t) &= - \int_{\Omega} \frac{1}{2} |\nabla u(\cdot, t)|^2 dx - \int_{\Omega} \frac{1}{2} |\nabla w(\cdot, t)|^2 dx \\ &- \int_{\Omega} \nabla u(\cdot, t) \cdot \nabla(\nabla \cdot \xi)(\cdot, t) dx \\ &+ \int_{\partial^* A(t) \cap \Omega} (B \cdot n_{\partial^* A})(\cdot, t) (w + (\nabla \cdot \xi))(\cdot, t) d\mathcal{H}^{d-1}, \end{aligned}$$

$$(30) \quad R_{\partial_t \xi}(t) = - \int_{\partial^* A(t) \cap \Omega} (\partial_t \xi + (B \cdot \nabla) \xi + (\nabla B)^\top \xi)(\cdot, t) \cdot (n_{\partial^* A} - \xi)(\cdot, t) d\mathcal{H}^{d-1}$$

$$\begin{aligned}
& - \int_{\partial^* A(t) \cap \Omega} (\partial_t \xi + (B \cdot \nabla) \xi)(\cdot, t) \cdot \xi(\cdot, t) d\mathcal{H}^{d-1}, \\
(31) \quad R_{\nabla B}(t) &= - \int_{\partial^* A(t) \cap \Omega} ((n_{\partial^* A} - \xi) \otimes (n_{\partial^* A} - \xi))(\cdot, t) : \nabla B(\cdot, t) d\mathcal{H}^{d-1} \\
& - \int_{\partial^* A(t) \cap \Omega} (n_{\partial^* A} \cdot \xi - 1)(\cdot, t) (\nabla \cdot B)(\cdot, t) d\mathcal{H}^{d-1},
\end{aligned}$$

$$\begin{aligned}
(32) \quad R_{\text{varifold}/BV}(t) &= - \int_{\partial^* A(t) \cap \Omega} (\text{Id} - n_{\partial^* A} \otimes n_{\partial^* A})(\cdot, t) : \nabla B(\cdot, t) d\mathcal{H}^{d-1} \\
& + \int_{\overline{\Omega} \times \mathbb{S}^{d-1}} (\text{Id} - \mathfrak{p} \otimes \mathfrak{p}) : \nabla B(\cdot, t) d\mu_t(\cdot, \mathfrak{p})
\end{aligned}$$

as well as

$$\begin{aligned}
(33) \quad U_{\text{dissip}}(t) &= - \int_{\Omega} \nabla u(\cdot, t) \cdot \nabla \vartheta(\cdot, t) dx \\
& + \int_{\partial^* A(t) \cap \Omega} \vartheta(\cdot, t) (n_{\partial^* A} \cdot B)(\cdot, t) d\mathcal{H}^{d-1},
\end{aligned}$$

$$(34) \quad U_{\partial_t \vartheta}(t) = \int_{\Omega} (\chi_A - \chi_{\mathcal{A}})(\cdot, t) (\partial_t \vartheta + (B \cdot \nabla) \vartheta)(\cdot, t) dx,$$

$$(35) \quad U_{\nabla B}(t) = \int_{\Omega} (\chi_A - \chi_{\mathcal{A}})(\cdot, t) \vartheta(\cdot, t) (\nabla \cdot B)(\cdot, t) dx.$$

3.2. Time splitting argument: Definition of good and bad times. In order to suitably estimate the right hand sides of (27) and (28), respectively, we essentially aim to reduce the task to a perturbative setting, where the interface of the weak solution can be represented as a graph over the interface of the smooth solution (with arbitrarily small C^1 norm). This reduction argument is implemented based on a case distinction for times $t \in (0, T)$, with the non-perturbative regime corresponding to either disproportionally large dissipation of energy of the weak solution or a lower bound for the error $E[A, \mu | \mathcal{A}]$. We formalize this as follows.

Let $\Lambda \in (0, \infty)$ and $M \in (1, \infty)$ constants we will determine later. We then construct a disjoint decomposition

$$(36) \quad (0, T_*) = \mathcal{T}_{\text{bad}}(\Lambda, M) \cup \mathcal{T}_{\text{good}}(\Lambda, M)$$

by means of

$$(37) \quad \mathcal{T}_{\text{bad}}(\Lambda, M) := \mathcal{T}_{\text{bad}}^{(1)}(\Lambda) \cup \mathcal{T}_{\text{bad}}^{(2)}(\Lambda, M),$$

$$(38) \quad \mathcal{T}_{\text{bad}}^{(1)}(\Lambda) := \left\{ t \in (0, T_*) : \int_{\Omega} \frac{1}{2} |\nabla w(\cdot, t)|^2 dx > \Lambda \right\},$$

$$(39) \quad \mathcal{T}_{\text{bad}}^{(2)}(\Lambda, M) := \left\{ t \in (0, T_*) \setminus \mathcal{T}_{\text{bad}}^{(1)}(\Lambda) : E[A, \mu | \mathcal{A}](t) > \frac{\ell(t)^{d-1}}{M} \right\},$$

$$(40) \quad \mathcal{T}_{\text{good}}(\Lambda, M) := (0, T_*) \setminus \mathcal{T}_{\text{bad}}(\Lambda, M).$$

The merit of these definitions is that on one side, the estimate for the right hand sides of (27) and (28) is easily closed for times $t \in \mathcal{T}_{\text{bad}}(\Lambda, M)$, whereas on the other side for times $t \in \mathcal{T}_{\text{good}}(\Lambda, M)$ one can actually *prove* that one has to be in a perturbative setting; cf. Proposition 5 below. The latter is a powerful tool to estimate the (not yet fully unraveled) “non-local contributions” in the right hand

sides of (27) and (28) originating from the gradient flow structure $(V_{\partial A}, H_{\partial A}) \in H_{MS}^{-1/2}(\partial A \cap \Omega) \times H_{MS}^{1/2}(\partial A \cap \Omega)$ of Mullins–Sekerka flow (1a)–(1d).

3.3. Stability estimates at bad times. In a first step, we will now estimate the right hand sides of (27) and (28) in the non-perturbative setting.

Lemma 4 (Stability estimates at bad times). *For all $T' \in (0, T_*)$ there exists $\Lambda = \Lambda(\mathcal{A}, A(0), T') \in (0, \infty)$ such that for all $M \in (1, \infty)$ and all $t \in \mathcal{T}_{\text{bad}}(\Lambda, M) \cap (0, T')$, it holds*

$$(41) \quad \begin{aligned} & R_{\text{dissip}}(t) + R_{\partial_t \xi}(t) + R_{\nabla B}(t) + R_{\text{vari-fold/BV}}(t) \\ & \leq \begin{cases} -\int_{\Omega} \frac{3}{8} |\nabla u(\cdot, t)|^2 + \frac{3}{8} |\nabla w(\cdot, t)|^2 dx, & t \in \mathcal{T}_{\text{bad}}^{(1)}(\Lambda), \\ -\int_{\Omega} \frac{3}{8} |\nabla u(\cdot, t)|^2 + \frac{1}{2} |\nabla w(\cdot, t)|^2 dx + \frac{\Lambda}{4} \frac{M}{\ell(t)^{d-1}} E[A, \mu | \mathcal{A}](t), & \text{else,} \end{cases} \end{aligned}$$

and

$$(42) \quad \begin{aligned} & U_{\text{dissip}}(t) + U_{\partial_t v}(t) + U_{\nabla B}(t) \\ & \leq \begin{cases} \int_{\Omega} \frac{1}{8} |\nabla u(\cdot, t)|^2 + \frac{1}{8} |\nabla w(\cdot, t)|^2 dx, & t \in \mathcal{T}_{\text{bad}}^{(1)}(\Lambda), \\ \int_{\Omega} \frac{1}{8} |\nabla u(\cdot, t)|^2 dx + \frac{\Lambda}{4} \frac{M}{\ell(t)^{d-1}} E[A, \mu | \mathcal{A}](t), & \text{else.} \end{cases} \end{aligned}$$

3.4. Stability estimates at good times. As already said, a key ingredient for our approach to weak-strong uniqueness and stability of the Mullins–Sekerka problem (1a)–(1d) consists of a reduction argument to the interface of the weak solution being given as a graph over the interface of the smoothly evolving solution with arbitrarily small C^1 norm. It is a non-trivial result that our definition (40) of good times is actually a sufficient condition to imply such a perturbative graph setting, an implication which we formalize in the following result.

Proposition 5 (Perturbative graph setting at good times). *Fix $T' \in (0, T_*)$, and let $\Lambda = \Lambda(\mathcal{A}, A(0), T') \in (0, \infty)$ be the constant from Lemma 4. For every $C \in (1, \infty)$ there exists a constant*

$$(43) \quad M = M(\Omega, A(0), \mathcal{A}, C, \Lambda, T') \in (1, \infty)$$

such that for almost every $t \in \mathcal{T}_{\text{good}}(\Lambda, M) \cap (0, T')$ there exists a height function

$$(44) \quad h(\cdot, t) \in (C^{1,1-\frac{d-1}{4}} \cap H^2)(\partial \mathcal{A}(t))$$

such that

$$(45) \quad \partial^* A(t) = \{x + h(x, t) \mathbf{n}_{\partial \mathcal{A}}(x, t) : x \in \partial \mathcal{A}(t)\} \subset \{x \in \Omega : \text{dist}(x, \partial \Omega) > \ell(t)\},$$

$$(46) \quad |\mu_t|_{\mathbb{S}^{d-1}} = \mathcal{H}^{d-1} \llcorner \partial^* A(t),$$

and

$$(47) \quad \frac{1}{\ell(t)} \|h(\cdot, t)\|_{L^\infty(\partial \mathcal{A}(t))} + \|\nabla_{\partial \mathcal{A}} h(\cdot, t)\|_{L^\infty(\partial \mathcal{A}(t))} \leq \frac{1}{16C}.$$

We continue on how to close the stability estimate for times at which the previous result holds, treating it as a black-box. The first essential step consists of suitably rewriting the terms constituting R_{dissip} and U_{dissip} from (29) and (33), respectively.

To this end, one may draw inspiration from the gradient flow structure of Mullins–Sekerka flow (1a)–(1d), which, for the smoothly evolving $\partial \mathcal{A}$, is given

as follows. Denote by $H_{MS}^{1/2}(\partial\mathcal{A})$ the set of all maps $f: \partial\mathcal{A} \rightarrow \mathbb{R}$ arising as traces of maps from $H^1(\mathcal{A})$ such that $\int_{\partial\mathcal{A}} f d\mathcal{H}^{d-1} = 0$. One then defines a Hilbert space structure on $H_{MS}^{1/2}(\partial\mathcal{A})$ by means of the minimal Dirichlet energy extension to Ω :

$$(48) \quad \langle f, \tilde{f} \rangle_{H_{MS}^{1/2}(\partial\mathcal{A})} := \int_{\Omega} \nabla u_f \cdot \nabla u_{\tilde{f}} dx,$$

where, for given $f \in H_{MS}^{1/2}(\partial\mathcal{A})$, the associated potential $u_f \in H^1(\Omega)$ satisfies

$$\begin{aligned} \Delta u_f &= 0 && \text{in } \Omega \setminus \partial\mathcal{A}, \\ u_f &= f && \text{on } \partial\mathcal{A}, \\ (\mathbf{n}_{\partial\Omega} \cdot \nabla) u_f &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Denoting by $H_{MS}^{-1/2}(\partial\mathcal{A})$ the dual of $H_{MS}^{1/2}(\partial\mathcal{A})$, it is immediate to recognize that the Riesz isomorphism $\mathcal{R}: H_{MS}^{-1/2}(\partial\mathcal{A}) \rightarrow H_{MS}^{1/2}(\partial\mathcal{A})$ is formally realized through the inverse of the (two-phase) Dirichlet-to-Neumann operator:

$$\langle F, f \rangle_{H_{MS}^{-1/2}(\partial\mathcal{A}) \times H_{MS}^{1/2}(\partial\mathcal{A})} = \langle \mathcal{R}(F), f \rangle_{H_{MS}^{1/2}(\partial\mathcal{A})} = - \int_{\partial\mathcal{A}} (\mathbf{n}_{\partial\mathcal{A}} \cdot \llbracket \nabla u_{\mathcal{R}(F)} \rrbracket) f d\mathcal{H}^{d-1},$$

as can be seen similarly to the computation (3). Note that the integration by parts is rigorous for those F such that $\mathcal{R}(F)$ is the trace of a, say, $H^2(\mathcal{A})$ function.

The idea for the rewriting of, say, R_{dissip} is now as follows. What one likes to achieve in (29) is to complete squares in the two quadratic terms originating from the energy dissipation inequality of a weak solution such that the resulting quadratic terms penalize the *difference of curvatures* of the two evolving geometries as measured through the correct gradient flow norm. This is indeed in direct analogy of how we proceeded in the case of (multiphase) mean curvature flow being the $L^2(\partial\mathcal{A})$ gradient flow of the perimeter; cf. Section 3.1 of [9]. The next result implements this procedure for Mullins–Sekerka flow.

Lemma 6 (Structure of dissipative terms in perturbative regime). *Fix $t \in (0, T_*)$, and assume that for t the conclusions (44)–(47) of Proposition 5 hold true. Furthermore, assume that t is a Lebesgue point for $s \mapsto u(\cdot, s)$ in the sense that*

$$(49) \quad \lim_{\tau \searrow 0} \int_{t-\tau}^{t+\tau} \int_{\Omega} |\nabla u - \nabla u(\cdot, t)|^2 dx ds = 0.$$

i) Defining a chemical potential $\tilde{w} = \tilde{w}(\cdot, t) \in H^1(\Omega)$ by means of

$$(50) \quad -\Delta \tilde{w} = 0 \quad \text{in } \Omega \setminus \partial^* A(t),$$

$$(51) \quad \text{tr}_{\partial^* A(t)} \tilde{w} = -(\nabla \cdot \xi)(\cdot, t) \quad \text{on } \partial^* A(t) \cap \Omega,$$

$$(52) \quad (\mathbf{n}_{\partial\Omega} \cdot \nabla) \tilde{w} = 0 \quad \text{on } \partial\Omega,$$

we obtain the following alternative representation for R_{dissip} from (29):

$$(53) \quad \begin{aligned} R_{dissip}(t) &= - \int_{\Omega} \frac{1}{2} |(\nabla u - \nabla \tilde{w})(\cdot, t)|^2 dx - \int_{\Omega} \frac{1}{2} |(\nabla w - \nabla \tilde{w})(\cdot, t)|^2 dx \\ &\quad + \int_{\partial^* A(t) \cap \Omega} (w - \tilde{w})(\cdot, t) ((B \cdot \mathbf{n}_{\partial^* A}) + \llbracket (\mathbf{n}_{\partial^* A} \cdot \nabla) \tilde{w} \rrbracket)(\cdot, t) d\mathcal{H}^{d-1}. \end{aligned}$$

ii) Similarly, by defining another chemical potential $\tilde{w}_\vartheta = \tilde{w}_\vartheta(\cdot, t)$ through the problem

$$(54) \quad -\Delta \tilde{w}_\vartheta = 0 \quad \text{in } \Omega \setminus \partial^* A(t),$$

$$(55) \quad \text{tr}_{\partial^* A(t)} \tilde{w}_\vartheta = \vartheta(\cdot, t) \quad \text{on } \partial^* A(t) \cap \Omega,$$

$$(56) \quad (\mathbf{n}_{\partial\Omega} \cdot \nabla) \tilde{w}_\vartheta = 0 \quad \text{on } \partial\Omega,$$

we obtain that U_{dissip} from (33) can be rewritten as follows:

$$(57) \quad U_{dissip}(t) = \int_{\partial^* A(t) \cap \Omega} \vartheta(\cdot, t) ((B \cdot \mathbf{n}_{\partial^* A}) + [(\mathbf{n}_{\partial^* A} \cdot \nabla) \tilde{w}]) (\cdot, t) d\mathcal{H}^{d-1} \\ + \int_{\partial^* A(t) \cap \Omega} (u - \tilde{w})(\cdot, t) [(\mathbf{n}_{\partial^* A} \cdot \nabla) \tilde{w}_\vartheta] (\cdot, t) d\mathcal{H}^{d-1}.$$

The second, and most straightforward, step is to estimate the terms on the right hand sides of (27) and (28) that only feature local terms. That is, they only depend on the kinematics of the problem and do not need to be treated using the nonlocal gradient flow structure of Mullins-Sekerka flow via the dissipative terms from (53).

Lemma 7 (Stability estimate for local terms in perturbative regime). *Fix $t \in (0, T_*)$, and assume that for t the conclusions (44)–(47) of Proposition 5 hold true. For any collection (ξ, ϑ, B) satisfying only the assumptions (11)–(16) and (24)–(26), we have the estimates*

$$(58) \quad R_{\partial_t \xi}(t) = 0,$$

$$(59) \quad R_{\nabla B}(t) \leq \|(\nabla \cdot B)(\cdot, t)\|_{L^\infty(B_{\ell(t)/2}(\partial\mathcal{A}(t)))} E_{rel}[A, \mu|\mathcal{A}](t) \\ + \|(\nabla B)(\cdot, t) + (\nabla B)^\top(\cdot, t)\|_{L^\infty(B_{\ell(t)/2}(\partial\mathcal{A}(t)))} E_{rel}[A, \mu|\mathcal{A}](t),$$

$$(60) \quad R_{\text{varifold}/BV}(t) \leq \|(\nabla \cdot B)(\cdot, t)\|_{L^\infty(B_{\ell(t)/2}(\partial\mathcal{A}(t)))} E_{rel}[A, \mu|\mathcal{A}](t) \\ + 4\|(\nabla B)(\cdot, t) + (\nabla B)^\top(\cdot, t)\|_{L^\infty(B_{\ell(t)/2}(\partial\mathcal{A}(t)))} E_{rel}[A, \mu|\mathcal{A}](t),$$

as well as

$$(61) \quad U_{\partial_t \vartheta}(t) \leq \frac{|\partial_t \ell(t)|}{\ell(t)} E_{vol}[A|\mathcal{A}](t),$$

$$(62) \quad U_{\nabla B}(t) \leq \|(\nabla \cdot B)(\cdot, t)\|_{L^\infty(B_{\ell(t)/2}(\partial\mathcal{A}(t)))} E_{vol}[A|\mathcal{A}](t).$$

In a third and final step, we have to provide an estimate for the remaining terms given by (53) and (57). Focusing for the moment on R_{dissip} , the idea again draws heavily from the gradient flow structure of Mullins-Sekerka flow and goes as follows.

First, as already discussed above, the motivation for the argument leading to (53) is that $-\int_\Omega \frac{1}{2} |\nabla(w - \tilde{w})|^2 dx$ penalizes the difference of curvatures of the two solutions as measured through the gradient flow structure associated with Mullins-Sekerka flow. Indeed, $-\nabla \cdot \xi$ serves as a proxy for $H_{\partial\mathcal{A}}$ away from $\partial\mathcal{A}$ since ξ represents an extension of the unit normal vector field $\mathbf{n}_{\partial\mathcal{A}}$, and $w - \tilde{w}$ is by construction of \tilde{w} and the Gibbs-Thomson law (7e) the minimal Dirichlet extension of $H_{\partial^* A \cap \Omega} + \nabla \cdot \xi$ away from the interface $\partial^* A \cap \Omega$. In the perturbative graph setting of Proposition 5, one may thus expect that to leading order (and up to

terms controlled by the interface error $E[A, \mu|\mathcal{A}]$)

$$(63) \quad - \int_{\Omega} \frac{1}{2} |\nabla(w - \tilde{w})|^2 dx \approx -\frac{1}{2} \|\Delta_{\partial\mathcal{A}} h\|_{H_{MS}^{1/2}(\partial\mathcal{A})}^2 = -\frac{1}{2} \|\nabla u_{\Delta_{\partial\mathcal{A}} h}\|_{L^2(\Omega)}^2,$$

where as before $u_f \in H^1(\Omega)$ denotes the minimal Dirichlet energy extension of $f \in H_{MS}^{1/2}(\partial\mathcal{A})$ to Ω .

Second, since B is an extension of the normal velocity of the smoothly evolving $\partial\mathcal{A}$ (i.e., an extension of the jump of the Neumann data of the chemical potential of the strong solution, the latter being equal to $H_{\partial\mathcal{A}}$ along $\partial\mathcal{A}$), in the perturbative graph setting of Proposition 5, one should think of the term $\mathbf{n}_{\partial^* A} \cdot (B + \llbracket \nabla \tilde{w} \rrbracket)$ along $\partial^* \mathcal{A}$ as a proxy for $\mathbf{n}_{\partial\mathcal{A}} \cdot \llbracket \nabla u_{(-\text{tr}(L_{\partial\mathcal{A}}^2)h - f_{\partial\mathcal{A}} - \text{tr}(L_{\partial\mathcal{A}}^2)h) d\mathcal{H}^{d-1}} \rrbracket$ along $\partial\mathcal{A}$, since to leading order the associated Dirichlet data is given by $H_{\partial\mathcal{A}} + \nabla \cdot \xi|_{\partial^* A \cap \Omega} \approx -\text{tr}(L_{\partial\mathcal{A}}^2)h$, where we denote by $L_{\partial\mathcal{A}}$ the Weingarten tensor of $\partial\mathcal{A}$ (see, e.g., [25, Equation (2.7), page 47]). Hence, by the very definition (48) of the Hilbert space structure on $H_{MS}^{1/2}(\partial\mathcal{A})$ one may expect that to leading order (and up to terms controlled by the interface error $E[A, \mu|\mathcal{A}]$)

$$(64) \quad \begin{aligned} & \int_{\partial^* A(t) \cap \Omega} (w - \tilde{w})(\cdot, t) ((B \cdot \mathbf{n}_{\partial^* A}) + \llbracket (\mathbf{n}_{\partial^* A} \cdot \nabla) \tilde{w} \rrbracket)(\cdot, t) d\mathcal{H}^{d-1} \\ & \approx \left\langle \Delta_{\partial\mathcal{A}} h, -\text{tr}(L_{\partial\mathcal{A}}^2)h - \int_{\partial\mathcal{A}} -\text{tr}(L_{\partial\mathcal{A}}^2)h d\mathcal{H}^{d-1} \right\rangle_{H_{MS}^{1/2}(\partial\mathcal{A})} \\ & = \int_{\Omega} \nabla u_{\Delta_{\partial\mathcal{A}} h} \cdot \nabla u_{(-\text{tr}(L_{\partial\mathcal{A}}^2)h - f_{\partial\mathcal{A}} - \text{tr}(L_{\partial\mathcal{A}}^2)h) d\mathcal{H}^{d-1}} dx. \end{aligned}$$

Hence, in view of the representation (53) for R_{dissip} , one may expect in the perturbative graph setting of Proposition 5 to obtain to leading order (and up to terms controlled by the interface error $E[A, \mu|\mathcal{A}]$) an estimate of the form

$$(65) \quad \begin{aligned} R_{dissip} & \approx - \int_{\Omega} \frac{1}{2} |\nabla(u - \tilde{w})|^2 dx - \int_{\Omega} \frac{3}{8} |\nabla(w - \tilde{w})|^2 dx \\ & + \tilde{C} \left\| -\text{tr}(L_{\partial\mathcal{A}}^2)h - \int_{\partial\mathcal{A}} -\text{tr}(L_{\partial\mathcal{A}}^2)h d\mathcal{H}^{d-1} \right\|_{H_{MS}^{1/2}(\partial\mathcal{A})}^2 \end{aligned}$$

for some constant $\tilde{C} > 0$.

Finally, in the perturbative graph setting of Proposition 5, it is not hard to show (by a change to tubular neighborhood coordinates for E_{vol} , and by expressing $\mathbf{n}_{\partial^* A} \cdot \xi$ to leading order in terms of the graph function h and its derivative h' for E_{rel}) that our error functional behaves to leading order as

$$(66) \quad E[A, \mu|\mathcal{A}] \approx \left(\frac{1}{2} \left\| \frac{h}{\ell} \right\|_{L^2(\partial\mathcal{A})}^2 + \frac{1}{2} \|h'\|_{L^2(\partial\mathcal{A})}^2 \right).$$

Hence, once one is provided with an interpolation estimate of the form

$$(67) \quad \|f\|_{H_{MS}^{1/2}(\partial\mathcal{A})}^2 \lesssim \|f\|_{H^1(\partial\mathcal{A})} \|f\|_{H^2(\partial\mathcal{A})},$$

one may expect based on (65)–(67)

$$(68) \quad R_{dissip} \leq - \int_{\Omega} \frac{1}{2} |\nabla(u - \tilde{w})|^2 dx - \int_{\Omega} \frac{1}{4} |\nabla(w - \tilde{w})|^2 dx + \tilde{C} E[A, \mu|\mathcal{A}],$$

which renders the following rigorous version of this heuristic conceivable. Starting point for its proof will be a domain mapping argument (not surprisingly in the form of a Hanzawa transformation), cf. the more technical discussion of Subsection 5.5.

Proposition 8 (Stability estimates for non-local terms in perturbative regime). *Let $\delta \in (0, 1)$, fix $t \in (0, T_*)$, and assume that for t the conclusions (44)–(47) of Proposition 5 hold true. In particular, recall that for a given $C \in (1, \infty)$ it holds*

$$(69) \quad \frac{1}{\ell(t)} \|h(\cdot, t)\|_{L^\infty(\partial\mathcal{A}(t))} + \|\nabla_{\partial\mathcal{A}} h(\cdot, t)\|_{L^\infty(\partial\mathcal{A}(t))} \leq \frac{1}{16C}.$$

Furthermore, denote by $\Lambda \in (1, \infty)$ and $M \in (1, \infty)$ two constants such that

$$(70) \quad \int_{\Omega} \frac{1}{2} |\nabla w(\cdot, t)|^2 \leq \Lambda,$$

$$(71) \quad E[A, \mu|\mathcal{A}](t) \leq \frac{\ell(t)^{d-1}}{M}.$$

Finally, fix (ξ, ϑ, B) satisfying the assumptions (11)–(16) and (24)–(26), and let the chemical potentials \tilde{w} and \tilde{w}_ϑ be defined according to Lemma 6.

Then, one may choose $C \gg_\delta 1$ and $M \gg_\delta 1$ locally uniformly in $[0, T_*)$ such that there exists a constant $\tilde{C} = \tilde{C}(\mathcal{A}, \Omega, \Lambda, \ell, M, T^*) \in (1, \infty)$ such that

$$(72) \quad \begin{aligned} & \int_{\partial^* A(t) \cap \Omega} (w - \tilde{w})(\cdot, t) ((B \cdot \mathbf{n}_{\partial^* A}) + [(\mathbf{n}_{\partial^* A} \cdot \nabla) \tilde{w}])(\cdot, t) d\mathcal{H}^{d-1} \\ & \leq \delta \int_{\Omega} |\nabla(w - \tilde{w})(\cdot, t)|^2 dx + \frac{\tilde{C}}{\delta} E[A, \mu|\mathcal{A}](t), \end{aligned}$$

and

$$(73) \quad \begin{aligned} & \int_{\partial^* A(t) \cap \Omega} (u - \tilde{w})(\cdot, t) [(\mathbf{n}_{\partial^* A} \cdot \nabla) \tilde{w}_\vartheta](\cdot, t) d\mathcal{H}^{d-1} \\ & \leq \delta \int_{\Omega} |\nabla(u - \tilde{w})(\cdot, t)|^2 dx + \delta \int_{\Omega} |\nabla(w - \tilde{w})(\cdot, t)|^2 dx + \frac{\tilde{C}}{\delta} E[A, \mu|\mathcal{A}](t), \end{aligned}$$

as well as

$$(74) \quad \begin{aligned} & \int_{\partial^* A(t) \cap \Omega} \vartheta(\cdot, t) ((B \cdot \mathbf{n}_{\partial^* A}) + [(\mathbf{n}_{\partial^* A} \cdot \nabla) \tilde{w}])(\cdot, t) d\mathcal{H}^{d-1} \\ & \leq \delta \int_{\Omega} |\nabla(w - \tilde{w})(\cdot, t)|^2 dx + \frac{\tilde{C}}{\delta} E[A, \mu|\mathcal{A}](t), \end{aligned}$$

4. PROOF OF THEOREM 1

We proceed in three steps.

Step 1: Construction of triple (ξ, ϑ, B) satisfying (11)–(16) and (24)–(26). Let $\bar{\eta} \in C_{cpt}^\infty(\mathbb{R}; [0, 1])$ be such that $\text{supp } \bar{\eta} \subset [-3/4, 3/4]$, $\bar{\eta} \equiv 1$ on $[-1/2, 1/2]$ and $|\bar{\eta}'| \leq 8$. We then define the vector fields ξ and B by means of

$$(75) \quad \xi(x, t) := \bar{\eta} \left(\frac{s_{\partial\mathcal{A}}(x, t)}{\ell(t)} \right) \nabla s_{\partial\mathcal{A}}(x, t),$$

$$(76) \quad B(x, t) := \bar{\eta} \left(\frac{s_{\partial\mathcal{A}}(x, t)}{\ell(t)} \right) \mathbf{V}_{\partial\mathcal{A}}(P_{\partial\mathcal{A}}(x, t), t),$$

where $(x, t) \in \bar{\Omega} \times [0, T_*)$. We further choose $\bar{\vartheta} \in C^\infty(\mathbb{R}; [-1, 1])$ such that $\bar{\vartheta}(s) = s$ on $[-1/2, 1/2]$, $\bar{\vartheta} \equiv -1$ on $(-\infty, -1]$, $\bar{\vartheta} \equiv 1$ on $[1, \infty)$, and finally $\bar{\vartheta}' > 0$ in $(-1, 1)$. Based on this auxiliary map, we define the weight ϑ as follows:

$$(77) \quad \vartheta(x, t) := -\frac{1}{\ell(t)} \bar{\vartheta}\left(\frac{s_{\partial\mathcal{A}}(x, t)}{\ell(t)}\right),$$

where $(x, t) \in \bar{\Omega} \times [0, T_*)$. The properties (11)–(16) and (24)–(26) are immediate consequences of the definitions (75)–(77) and the regularity of the smoothly evolving phase \mathcal{A} .

Step 2: Proof of stability estimate (5). Given the triple (ξ, ϑ, B) from the previous step, define the functional $(0, T_*) \ni t \mapsto E[A, \mu|\mathcal{A}](t) \in L^\infty_{loc}((0, T_*); [0, \infty))$ by (19).

Starting point for the proof of the stability estimate (5) are the preliminary stability estimates (27) and (28) from Lemma 3. Post-processing these by means of Lemma 4, Proposition 5, Lemma 6, Lemma 7, and Proposition 8, we infer that for almost every $T' \in (0, T_*)$ there is $C(T') > 0$ such that

$$(78) \quad \begin{aligned} & E[A, \mu|\mathcal{A}](T') + \int_0^{T'} \chi_{\mathcal{T}_{\text{bad}}} \left(\int_{\Omega} \frac{1}{4} |\nabla u|^2 dx + \int_{\Omega} \frac{1}{4} |\nabla w|^2 dx \right) dt \\ & + \int_0^{T'} \chi_{\mathcal{T}_{\text{good}}} \left(\int_{\Omega} \frac{1}{4} |(\nabla u - \nabla \tilde{w})|^2 dx + \int_{\Omega} \frac{1}{4} |(\nabla w - \nabla \tilde{w})|^2 dx \right) dt \\ & \leq E[A, \mu|\mathcal{A}](0) + C(T') \int_0^{T'} E[A, \mu|\mathcal{A}](t) dt. \end{aligned}$$

This of course implies the claim (5).

Step 3: Weak-strong uniqueness. By Grönwall's inequality, it follows from the weak-strong stability estimate (5) that

$$E[A, \mu|\mathcal{A}](T') \leq E[A, \mu|\mathcal{A}](0) e^{\int_0^{T'} C(t) dt}$$

for almost every $T' \in (0, T_*)$. Weak-strong uniqueness in the form of (6) thus follows from (4). \square

5. WEAK-STRONG STABILITY ESTIMATES FOR MULLINS-SEKERKA FLOW

5.1. Proof of Lemma 3: Preliminary relative entropy inequality. We proceed in two steps.

Step 1: Control by the dissipation estimate and by testing the weak formulation. Starting point for the derivation of (27) is the following representation of the relative entropy:

$$E_{\text{rel}}[A, \mu|\mathcal{A}](T') = E[\mu_{T'}] + \int_{A(T')} (\nabla \cdot \xi)(\cdot, T') dx$$

for all $T' \in [0, T_*)$, where we integrated by parts in the definition (17) of the relative entropy and used that $\xi(\cdot, T')$ is compactly supported within Ω , see (16). We thus infer from the dissipation estimate (7f) and the weak formulation (7b) applied to $\nabla \cdot \xi$ that

$$(79) \quad \begin{aligned} & E_{\text{rel}}[A, \mu|\mathcal{A}](T') \\ & \leq E_{\text{rel}}[A, \mu|\mathcal{A}](0) - \int_0^{T'} \int_{\Omega} \frac{1}{2} |\nabla u|^2 dx dt - \int_0^{T'} \int_{\Omega} \frac{1}{2} |\nabla w|^2 dx dt \end{aligned}$$

$$- \int_0^{T'} \int_{\Omega} \nabla u \cdot \nabla (\nabla \cdot \xi) \, dx dt - \int_0^{T'} \int_{\partial^* A(t) \cap \Omega} \partial_t \xi \cdot \mathbf{n}_{\partial^* A} \, d\mathcal{H}^{d-1} dt$$

for a.e. $T' \in [0, T_*)$.

Starting point for the proof of (28) is in turn the observation

$$E_{vol}[A|\mathcal{A}](T') = \int_{\Omega} (\chi_A - \chi_{\mathcal{A}})(\cdot, T') \vartheta(\cdot, T') \, dx$$

due to the sign conditions (15). As ϑ vanishes on $\partial\mathcal{A}$ by (14), we may use the the weak formulation (7b) applied to the strong solution and ϑ , along with an integration by parts in the potential term, to get

$$\int_0^{T'} \int_{\Omega} \chi_{\mathcal{A}} \partial_t \vartheta \, dx dt = 0,$$

so that the weak formulation (7b) applied to the weak solution gives

$$(80) \quad E_{vol}[A|\mathcal{A}](T') = E_{vol}[A|\mathcal{A}](0) + \int_0^{T'} \int_{\Omega} (\chi_A - \chi_{\mathcal{A}}) \partial_t \vartheta \, dx dt \\ - \int_0^{T'} \int_{\Omega} \nabla u \cdot \nabla \vartheta \, dx dt.$$

Step 2: Inserting expected PDEs satisfied by $\partial_t \xi$ and $\partial_t \vartheta$. Since the vector fields $\xi(\cdot, t)$ and $B(\cdot, t)$ are compactly supported within Ω for all $t \in [0, T_*)$, see (16) and (25), by now standard computations for the relative entropy method in curvature driven interface evolution yield (cf. [13, identity (2.11)])

$$(81) \quad - \int_0^{T'} \int_{\partial^* A(t) \cap \Omega} \partial_t \xi \cdot \mathbf{n}_{\partial^* A} \, d\mathcal{H}^{d-1} dt \\ = - \int_0^{T'} \int_{\partial^* A(t) \cap \Omega} (\text{Id} - \mathbf{n}_{\partial^* A} \otimes \mathbf{n}_{\partial^* A}) : \nabla B \, d\mathcal{H}^{d-1} dt \\ + \int_0^{T'} \int_{\partial^* A(t) \cap \Omega} (B \cdot \mathbf{n}_{\partial^* A}) (\nabla \cdot \xi) \, d\mathcal{H}^{d-1} dt \\ - \int_0^{T'} \int_{\partial^* A(t) \cap \Omega} (\partial_t \xi + (B \cdot \nabla) \xi + (\nabla B)^\top \xi) \cdot (\mathbf{n}_{\partial^* A} - \xi) \, d\mathcal{H}^{d-1} dt \\ - \int_0^{T'} \int_{\partial^* A(t) \cap \Omega} (\partial_t \xi + (B \cdot \nabla) \xi) \cdot \xi \, d\mathcal{H}^{d-1} dt \\ - \int_0^{T'} \int_{\partial^* A(t) \cap \Omega} (\mathbf{n}_{\partial^* A} - \xi) \otimes (\mathbf{n}_{\partial^* A} - \xi) : \nabla B \, d\mathcal{H}^{d-1} dt \\ - \int_0^{T'} \int_{\partial^* A(t) \cap \Omega} (\mathbf{n}_{\partial^* A} \cdot \xi - 1) (\nabla \cdot B) \, d\mathcal{H}^{d-1} dt$$

for a.e. $T' \in [0, T_*)$. Adding zero, then making use of the isotropic Gibbs–Thomson law (7e) and finally plugging everything back into (79) yields (27).

Next, adding and subtracting $(\chi_A - \chi_{\mathcal{A}})(B \cdot \nabla) \vartheta$, as well as integration by parts separately for both solutions and using the facts that ϑ vanishes on $\partial\mathcal{A}$ and that B

has compact support, as per identities (14) and (25), shows that

$$(82) \quad \int_0^{T'} \int_{\Omega} (\chi_A - \chi_{\mathcal{A}}) \partial_t \vartheta \, dx dt = \int_0^{T'} \int_{\Omega} (\chi_A - \chi_{\mathcal{A}}) (\partial_t \vartheta + (B \cdot \nabla) \vartheta) \, dx dt \\ + \int_0^{T'} \int_{\Omega} (\chi_A - \chi_{\mathcal{A}}) \vartheta \nabla \cdot B \, dx dt \\ + \int_0^{T'} \int_{\partial^* A(t) \cap \Omega} \vartheta \mathbf{n}_{\partial^* A} \cdot B \, d\mathcal{H}^{d-1} dt.$$

Plugging (82) back into (80) concludes the proof. \square

5.2. Proof of Lemma 4: Stability estimates at bad times. For notational simplicity, we drop the time dependence in all occurring terms. Looking back to the first step of the proof of Lemma 3 (or alternatively, simply choosing $B \equiv 0$ at bad times), we see that

$$(83) \quad R_{dissip} + R_{\partial_t \xi} + R_{\nabla B} = - \int_{\Omega} \frac{1}{2} |\nabla u|^2 \, dx - \int_{\Omega} \frac{1}{2} |\nabla w|^2 \, dx \\ - \int_{\Omega} \nabla u \cdot \nabla (\nabla \cdot \xi) \, dx \\ - \int_{\partial^* A \cap \Omega} \partial_t \xi \cdot \mathbf{n}_{\partial^* A} \, d\mathcal{H}^{d-1}, \\ (84) \quad U_{dissip} + U_{\partial_t \vartheta} + U_{\nabla B} = \int_{\Omega} (\chi_A - \chi_{\mathcal{A}}) \partial_t \vartheta \, dx - \int_{\Omega} \nabla u \cdot \nabla \vartheta \, dx.$$

Step 1: Estimate for times $\mathcal{I}_{\text{bad}}^{(1)}(\Lambda)$. We simply estimate

$$(85) \quad - \int_{\Omega} \nabla u \cdot \nabla (\nabla \cdot \xi) \, dx - \int_{\partial^* A \cap \Omega} \partial_t \xi \cdot \mathbf{n}_{\partial^* A} \, d\mathcal{H}^{d-1} \\ \leq \int_{\Omega} \frac{1}{8} |\nabla u|^2 \, dx + 2\mathcal{L}^d(\text{supp } \xi) \|\nabla (\nabla \cdot \xi)\|_{L^\infty(\bar{\Omega})}^2 + E[A(0)] \|\partial_t \xi\|_{L^\infty(\bar{\Omega})}$$

and

$$(86) \quad \int_{\Omega} (\chi_A - \chi_{\mathcal{A}}) \partial_t \vartheta \, dx - \int_{\Omega} \nabla u \cdot \nabla \vartheta \, dx \\ \leq \int_{\Omega} \frac{1}{8} |\nabla u|^2 \, dx + 2\mathcal{L}^d(\text{supp } \nabla \vartheta) \|\nabla \vartheta\|_{L^\infty(\bar{\Omega})}^2 + \mathcal{L}^d(\text{supp } \partial_t \vartheta) \|\partial_t \vartheta\|_{L^\infty(\bar{\Omega})}^2.$$

Hence, in view of (83) and (84), any $\Lambda > 0$ satisfying

$$\Lambda \geq 4 \operatorname{ess\,sup}_{t \in (0, T')} (2\mathcal{L}^d(\text{supp } \xi) \|\nabla (\nabla \cdot \xi)\|_{L^\infty(\bar{\Omega})}^2 + E[A(0)] \|\partial_t \xi\|_{L^\infty(\bar{\Omega})})(t), \\ \Lambda \geq 4 \operatorname{ess\,sup}_{t \in (0, T')} (2\mathcal{L}^d(\text{supp } \nabla \vartheta) \|\nabla \vartheta\|_{L^\infty(\bar{\Omega})}^2 + \mathcal{L}^d(\text{supp } \partial_t \vartheta) \|\partial_t \vartheta\|_{L^\infty(\bar{\Omega})}^2)(t)$$

does the job for (41) and (42) due to definition (38).

Step 2: Estimate for times $\mathcal{I}_{\text{bad}}^{(2)}(\Lambda, M)$. Just looking back at (83)–(86) and recalling the admissible choices for $\Lambda > 0$ from the previous display, we see that (41) and (42) are immediate consequences of definition (39). \square

5.3. Proof of Lemma 6: Structure of dissipative terms. Before we derive the representation formulas (53) and (57), we state two helpful intermediate results whose proofs are postponed until the end of this subsection. The first infers from the motion law (7b) harmonicity of the chemical potential $u(\cdot, t)$ away from the interface.

Lemma 9 (Harmonicity of the chemical potential in perturbative regime). *Fix $t \in (0, T_*)$ subject to the assumptions of Lemma 6. Then $u(\cdot, t)$ satisfies*

$$(87) \quad \int_{A(t)} \nabla u(\cdot, t) \cdot \nabla \zeta \, dx = 0$$

for all $\zeta \in C_{cpt}^\infty(A(t))$, and

$$(88) \quad \int_{\Omega \setminus A(t)} \nabla u(\cdot, t) \cdot \nabla \zeta \, dx = 0$$

for all $\zeta \in C_{cpt}^\infty(\overline{\Omega} \setminus \overline{A(t)})$.

The second auxiliary result exploits the perturbative setting to facilitate a rigorous integration by parts involving the jumps of the Neumann data for the chemical potentials along the interface $\partial^* A(t) \cap \Omega$, which in turn then allows, among other things, to “smuggle in” the chemical potential $\tilde{w}(\cdot, t)$ by means of its boundary condition (51).

Lemma 10 (Integration by parts formulas). *Fix $t \in (0, T_*)$ subject to the assumptions of Lemma 6. Then*

$$(89) \quad - \int_{\Omega} \nabla u(\cdot, t) \cdot \nabla(\nabla \cdot \xi)(\cdot, t) \, dx = \int_{\Omega} \nabla u(\cdot, t) \cdot \nabla \tilde{w}(\cdot, t) \, dx$$

and

$$(90) \quad \begin{aligned} & - \int_{\Omega} \nabla \tilde{w}(\cdot, t) \cdot \nabla(w - \tilde{w})(\cdot, t) \, dx \\ & = \int_{\partial^* A(t) \cap \Omega} (w - \tilde{w})(\cdot, t) [(\mathbf{n}_{\partial^* A} \cdot \nabla) \tilde{w}](\cdot, t) \, d\mathcal{H}^{d-1}, \end{aligned}$$

as well as

$$(91) \quad - \int_{\Omega} \nabla \tilde{w} \cdot \nabla \vartheta \, dx = \int_{\partial^* A \cap \Omega} \vartheta [(\mathbf{n}_{\partial^* A} \cdot \nabla) \tilde{w}] \, d\mathcal{H}^{d-1},$$

$$(92) \quad - \int_{\Omega} (\nabla u - \nabla \tilde{w}) \cdot \nabla \vartheta \, dx = \int_{\partial^* A \cap \Omega} (u - \tilde{w}) [(\mathbf{n}_{\partial^* A} \cdot \nabla) \tilde{w}_\vartheta] \, d\mathcal{H}^{d-1}.$$

We now combine the information from the previous two results to a proof of the representation formulas (53) and (57).

Proof of Lemma 6. The proof is split into two steps. For notational ease, we drop in the notation the time dependence of all quantities.

Step 1: Formula for R_{dissip} . For convenience, let us restate the definition of R_{dissip} from (29):

$$(93) \quad \begin{aligned} R_{dissip} &= - \int_{\Omega} \frac{1}{2} |\nabla u|^2 \, dx - \int_{\Omega} \frac{1}{2} |\nabla w|^2 \, dx - \int_{\Omega} \nabla u \cdot \nabla(\nabla \cdot \xi) \, dx \\ &\quad + \int_{\partial^* A \cap \Omega} (B \cdot \mathbf{n}_{\partial^* A})(w + (\nabla \cdot \xi)) \, d\mathcal{H}^{d-1}. \end{aligned}$$

Inserting the boundary condition (51), we observe that

$$(94) \quad \int_{\partial^* A \cap \Omega} (B \cdot \mathbf{n}_{\partial^* A})(w + (\nabla \cdot \xi)) d\mathcal{H}^{d-1} = \int_{\partial^* A \cap \Omega} (B \cdot \mathbf{n}_{\partial^* A})(w - \tilde{w}) d\mathcal{H}^{d-1}.$$

Next, by inserting the integration by parts formula (89) and completing squares, we get

$$(95) \quad \begin{aligned} & - \int_{\Omega} \frac{1}{2} |\nabla u|^2 dx - \int_{\Omega} \frac{1}{2} |\nabla w|^2 dx - \int_{\Omega} \nabla u \cdot \nabla(\nabla \cdot \xi) dx \\ & = - \int_{\Omega} \frac{1}{2} |\nabla u - \nabla \tilde{w}|^2 dx - \int_{\Omega} \frac{1}{2} |\nabla w - \nabla \tilde{w}|^2 dx - \int_{\Omega} \nabla \tilde{w} \cdot \nabla(w - \tilde{w}) dx. \end{aligned}$$

Hence, (53) follows from (93)–(95) and the integration by parts formula (90).

Step 2: Formula for U_{dissip} . Recall first from (33) that

$$(96) \quad U_{dissip} = - \int_{\Omega} \nabla u \cdot \nabla(\nabla \cdot \xi) dx + \int_{\partial^* A \cap \Omega} \vartheta(\mathbf{n}_{\partial^* A} \cdot B) d\mathcal{H}^{d-1}.$$

To upgrade this to (57), we simply start by adding zero in the form of

$$\begin{aligned} & - \int_{\Omega} \nabla u \cdot \nabla \vartheta dx + \int_{\partial^* A \cap \Omega} \vartheta \mathbf{n}_{\partial^* A} \cdot B d\mathcal{H}^{d-1} \\ & = - \int_{\Omega} (\nabla u - \nabla \tilde{w}) \cdot \nabla \vartheta dx - \int_{\Omega} \nabla \tilde{w} \cdot \nabla \vartheta dx + \int_{\partial^* A \cap \Omega} \vartheta \mathbf{n}_{\partial^* A} \cdot B d\mathcal{H}^{d-1}. \end{aligned}$$

Hence, (57) follows from the integration by parts formulas (91) and (92). \square

Proof of Lemma 9. Let $\zeta \in C_{cpt}^{\infty}(\bar{\Omega})$ be a nonnegative test function supported in $\bar{\Omega} \setminus \bar{A}(\bar{t})$. We then have by the nonnegativity of ζ , the property $\zeta \equiv 0$ in $A(t)$, and the weak formulation of the Mullins-Sekerka equation (7b)

$$0 \leq \int_{A(\bar{t})} \zeta dx = \int_{A(\bar{t})} \zeta dx - \int_{A(t)} \zeta dx \stackrel{(7b)}{=} - \int_t^{\bar{t}} \int_{\Omega} \nabla u \cdot \nabla \zeta dx ds.$$

Dividing by $|\bar{t} - t|$ and adding zero, we obtain

$$\begin{aligned} 0 & \leq - \text{sign}(\bar{t} - t) \int_{\Omega \setminus A(t)} \nabla u(\cdot, t) \cdot \nabla \zeta dx \\ & \quad - \frac{1}{|\bar{t} - t|} \int_t^{\bar{t}} \int_{\Omega} (\nabla u(\cdot, s) - \nabla u(\cdot, t)) \cdot \nabla \zeta dx ds. \end{aligned}$$

Passing to the limit $\bar{t} \searrow t$ respectively $\bar{t} \nearrow t$, we deduce by (49)

$$- \int_{\Omega \setminus A(t)} \nabla u(\cdot, t) \cdot \nabla \zeta dx \geq 0$$

respectively

$$- \int_{\Omega \setminus A(t)} \nabla u(\cdot, t) \cdot \nabla \zeta dx \leq 0.$$

This implies (88). Analogously, one shows (87) which then implies our lemma. \square

Proof of Lemma 10. We again drop the time dependence of all objects in the notation and proceed in three steps.

Step 1: Proof of (89). We note that due to (45) it holds $\partial^* A \cap \Omega = \partial^* A = \partial A \subset \subset \Omega$. Furthermore, Lemma 9 tells us that $\nabla \cdot \nabla u$ exists in the sense of square-integrable weak derivatives within the open sets A and $\Omega \setminus \bar{A}$, respectively, and that in fact

$$(97) \quad \nabla \cdot \nabla u = 0 \quad \text{throughout } A \text{ and } \Omega \setminus \bar{A}, \text{ respectively.}$$

In other words, $\nabla u \in Y_{div}^2(D) := \{f \in L^2(D; \mathbb{R}^d) : \nabla \cdot f \in L^2(D; \mathbb{R}^d)\}$ for $D \in \{A, \Omega \setminus \bar{A}\}$. It is a standard result (for instance in the context of mathematical fluid mechanics in connection with the Helmholtz decomposition; see, e.g., Simader and Sohr [28, Theorem 5.3]) that there exists a continuous linear operator $T_D : Y_{div}^2(D) \rightarrow W^{-\frac{1}{2}, 2}(\partial D)$ such that for all $f \in Y_{div}^2(D)$ and all $\zeta \in H^1(D)$

$$(98) \quad \int_D \zeta(\nabla \cdot f) dx = - \int_D \nabla \zeta \cdot f dx - \langle T_D(f), \zeta \rangle,$$

i.e., one may interpret $T_D f$ as $(n_{\partial D} \cdot f)|_{\partial D}$ in a weak sense. Denoting now by $\eta : \bar{\Omega} \rightarrow [0, 1]$ a C^1 -cutoff such that $\eta \equiv 1$ on \bar{A} and $\eta \equiv 0$ on $\partial\Omega$, we thus obtain from (97), (98) and using that ξ is compactly supported within Ω

$$(99) \quad \begin{aligned} & - \int_{\Omega} \nabla u \cdot \nabla(\nabla \cdot \xi) dx \\ &= - \int_A \nabla u \cdot \nabla(\nabla \cdot \xi) dx - \int_{\Omega \setminus \bar{A}} \nabla u \cdot \nabla(\eta + (1-\eta))(\nabla \cdot \xi) dx \\ &= \langle T_A(\nabla u), (\nabla \cdot \xi) \rangle + \langle T_{\Omega \setminus \bar{A}}(\nabla u), \eta(\nabla \cdot \xi) \rangle + \langle T_{\Omega \setminus \bar{A}}(\nabla u), (1-\eta)(\nabla \cdot \xi) \rangle. \end{aligned}$$

Since $(1-\eta) \equiv 0$ on ∂A by its choice and $\nabla \cdot \xi \equiv 0$ on $\partial\Omega$, it holds $(1-\eta)(\nabla \cdot \xi) \in H_0^1(\Omega \setminus \bar{D})$ and therefore

$$(100) \quad \langle T_{\Omega \setminus \bar{A}}(\nabla u), (1-\eta)(\nabla \cdot \xi) \rangle = 0.$$

Moreover, due to the boundary condition (51) and $\eta \equiv 0$ on $\partial\Omega$ by its choice, it holds $\nabla \cdot \xi = -\tilde{w} \in W^{\frac{1}{2}, 2}(\partial A)$ as well as $\eta \nabla \cdot \xi = -\eta \tilde{w} \in W^{\frac{1}{2}, 2}(\partial(\Omega \setminus \bar{A}))$ and therefore

$$\begin{aligned} & \langle T_A(\nabla u), \nabla \cdot \xi \rangle + \langle T_{\Omega \setminus \bar{A}}(\nabla u), \eta(\nabla \cdot \xi) \rangle \\ &= -\langle T_A(\nabla u), \tilde{w} \rangle - \langle T_{\Omega \setminus \bar{A}}(\nabla u), \eta \tilde{w} \rangle \\ &= -\langle T_A(\nabla u), \tilde{w} \rangle - \langle T_{\Omega \setminus \bar{A}}(\nabla u), \tilde{w} \rangle + \langle T_{\Omega \setminus \bar{A}}(\nabla u), (1-\eta)\tilde{w} \rangle. \end{aligned}$$

Based on (50) and (98), the previous display admits the upgrade

$$(101) \quad \begin{aligned} & \langle T_A(\nabla u), \nabla \cdot \xi \rangle + \langle T_{\Omega \setminus \bar{A}}(\nabla u), \eta(\nabla \cdot \xi) \rangle \\ &= \int_A \nabla u \cdot \nabla \tilde{w} dx + \int_{\Omega \setminus \bar{A}} \nabla u \cdot \nabla \tilde{w} dx + \langle T_{\Omega \setminus \bar{A}}(\nabla u), (1-\eta)\tilde{w} \rangle \\ &= \int_{\Omega} \nabla u \cdot \nabla \tilde{w} dx + \langle T_{\Omega \setminus \bar{A}}(\nabla u), (1-\eta)\tilde{w} \rangle. \end{aligned}$$

It remains to identify $\langle T_{\Omega \setminus \bar{A}}(\nabla u), (1-\eta)\tilde{w} \rangle$. However, comparing (98) applied to the data $(f, \zeta) = (\nabla u, (1-\eta)\tilde{w})$, $\nabla \cdot f = 0$, $D = \Omega \setminus \bar{A}$, with (88) applied to the admissible data $\zeta = (1-\eta)\tilde{w}$ yields (recall also that $\eta \equiv 0$ along $\partial\Omega$)

$$(102) \quad \langle T_{\Omega \setminus \bar{A}}(\nabla u), (1-\eta)\tilde{w} \rangle = 0.$$

The identities (99)–(102) together finally imply the claim (89).

Step 2: Proof of (90). Denote again by $\eta: \overline{\Omega} \rightarrow [0, 1]$ a C^1 -cutoff such that $\eta \equiv 1$ on \overline{A} and $\eta \equiv 0$ on $\partial\Omega$. Analogous to the above arguments, we then obtain on one side

$$-\int_{\Omega \setminus \overline{A}} \nabla \tilde{w} \cdot \nabla((1-\eta)(w - \tilde{w})) dx = \langle T_{\Omega \setminus \overline{A}}(\nabla \tilde{w}), (1-\eta)(w - \tilde{w}) \rangle = 0.$$

On the other side, we may exploit that the interface $\partial A = \partial^* A = \partial^* A \cap \Omega$ is $C^{1,\alpha}$ due to assumption (44). Since the boundary data in (51) is smooth, we get by standard Schauder theory that the auxiliary chemical potential is C^1 up to the interface ∂A from both sides, so that

$$\begin{aligned} & -\int_A \nabla \tilde{w} \cdot \nabla(w - \tilde{w}) dx - \int_{\Omega \setminus \overline{A}} \nabla \tilde{w} \cdot \nabla(\eta(w - \tilde{w})) dx \\ &= -\int_{\Omega} \nabla \tilde{w} \cdot \nabla(\eta(w - \tilde{w})) dx = \int_{\partial^* A \cap \Omega} [(\mathbf{n}_{\partial^* A} \cdot \nabla) \tilde{w}](w - \tilde{w}) d\mathcal{H}^{d-1}. \end{aligned}$$

The previous two displays obviously imply (90).

Step 3: Proof of (91) and (92). These two follow from analogous arguments. \square

5.4. Proof of Lemma 7: Stability estimate for local terms. Thanks to (47) and (45), we know $\partial^* A \cap \Omega \subset B_{\ell/4}(\partial\mathcal{A})$ as well as $A\Delta\mathcal{A} \subset B_{\ell/4}(\partial\mathcal{A})$. Hence, since $\frac{1}{2}|n-\xi|^2 = 1 - n \cdot \xi$ due to the first item of (13), we immediately obtain the estimates (59) and (62); recall for this also the definitions (17) and (18) of E_{rel} and E_{vol} , respectively. Furthermore, the identity (58) is a consequence of the following identities throughout $B_{\ell/4}(\partial\mathcal{A})$:

$$\begin{aligned} \xi \cdot (\partial_t \xi + (B \cdot \nabla) \xi) &= 0, \\ \partial_t \xi + (B \cdot \nabla) \xi + (\nabla B)^\top \xi &= 0. \end{aligned}$$

The first is true simply because of $|\xi| \equiv 1$ within $B_{\ell/2}(\partial\mathcal{A})$, see again the first item of (13). The second also follows from the latter by taking the spatial gradient of $\partial_t s_{\partial\mathcal{A}} + (B \cdot \nabla) s_{\partial\mathcal{A}} = 0$, which itself is true within $B_{\ell/2}(\partial\mathcal{A})$ thanks to (26). Note also in this context that the transport equation satisfied by the signed distance also directly implies (61) due to (14).

It remains to prove (60). To this end, we first obtain using (20) and (22)

$$\begin{aligned} & -\int_{\partial^* A \cap \Omega} \nabla \cdot B d\mathcal{H}^{d-1} \\ &= -\int_{\Omega} \varrho \nabla \cdot B d|\mu|_{\mathbb{S}^{d-1}} \leq -\int_{\overline{\Omega} \times \mathbb{S}^{d-1}} \nabla \cdot B d\mu + \|\nabla \cdot B\|_{L^\infty(B_{\ell/2}(\partial\mathcal{A}))} E_{rel}[A, \mu|_{\mathcal{A}}]. \end{aligned}$$

Adding zero several times, applying the compatibility condition from [15, Definition 3, item ii)] with test function $\eta = (\nabla B)^\top \xi$, and estimating based on the properties $(\xi \cdot \nabla) B = 0$ and $\frac{1}{2}|\xi - \mathbf{p}|^2 \leq 1 - \xi \cdot \mathbf{p}$, $\mathbf{p} \in \mathbb{S}^{d-1}$, valid throughout $B_{\ell/2}(\partial\mathcal{A})$, cf. (13) and (26), furthermore entails

$$\begin{aligned} \int_{\partial^* A \cap \Omega} \mathbf{n}_{\partial^* A} \otimes \mathbf{n}_{\partial^* A} : \nabla B d\mathcal{H}^{d-1} &= \int_{\partial^* A \cap \Omega} (\mathbf{n}_{\partial^* A} - \xi) \otimes \mathbf{n}_{\partial^* A} : \nabla B d\mathcal{H}^{d-1} \\ &\quad + \int_{\overline{\Omega} \times \mathbb{S}^{d-1}} \xi \otimes \mathbf{p} : \nabla B d\mu \\ &= \int_{\partial^* A \cap \Omega} (\mathbf{n}_{\partial^* A} - \xi) \otimes (\mathbf{n}_{\partial^* A} - \xi) : \nabla B d\mathcal{H}^{d-1} \end{aligned}$$

$$\begin{aligned}
& - \int_{\bar{\Omega} \times \mathbb{S}^{d-1}} (\mathbf{p} - \xi) \otimes (\mathbf{p} - \xi) : \nabla B \, d\mu \\
& + \int_{\bar{\Omega} \times \mathbb{S}^{d-1}} \mathbf{p} \otimes \mathbf{p} : \nabla B \, d\mu \\
\leq & \int_{\bar{\Omega} \times \mathbb{S}^{d-1}} \mathbf{p} \otimes \mathbf{p} : \nabla B \, d\mu \\
& + 4 \|\nabla B + (\nabla B)^\top\|_{L^\infty(B_{\ell/2}(\partial\mathcal{A}))} E_{rel}[A, \mu|_{\mathcal{A}}].
\end{aligned}$$

This concludes the proof. \square

5.5. Proof of Proposition 8: Stability estimates for non-local terms. For this whole subsection, let the assumptions and notation of Proposition 8 be in place. As before, let us drop the time dependence of all quantities in the notation. In order to streamline the proof of Proposition 8, we start by collecting an array of auxiliary constructions and results. Proofs for these are deferred until after we showed how the estimate (72) can be inferred from these. We finally conclude this subsection with the missing proofs for the estimates (73) and (74).

Throughout this subsection, we make use of the following convention: for a given point $y \in \partial\mathcal{A}$, we always denote by $(\mathbf{t}_{\partial\mathcal{A}}^i)_{i=1, \dots, d-1}$ an orthonormal basis for $\text{Tan}_y \partial\mathcal{A}$ such that each $\mathbf{t}_{\partial\mathcal{A}}^i$ represents a principal curvature direction at y with corresponding principal curvature $\kappa_{\partial\mathcal{A}}^i$. To avoid cumbersome notation in the following, let us also define

$$(103) \quad \bar{\mathbf{n}}_{\partial\mathcal{A}}(x) := \mathbf{n}_{\partial\mathcal{A}}(P_{\partial\mathcal{A}}(x)),$$

$$(104) \quad \bar{\mathbf{t}}_{\partial\mathcal{A}}^i(x) := \mathbf{t}_{\partial\mathcal{A}}^i(P_{\partial\mathcal{A}}(x)),$$

$$(105) \quad \bar{\kappa}_{\partial\mathcal{A}}^i(x) := (\kappa_{\partial\mathcal{A}}^i)(P_{\partial\mathcal{A}}(x)),$$

$$(106) \quad \bar{H}_{\partial\mathcal{A}}(x) := \sum_{i=1}^{d-1} \bar{\kappa}_{\partial\mathcal{A}}^i(x),$$

$$(107) \quad \bar{h}(x) := h(P_{\partial\mathcal{A}}(x))$$

for $x \in B_{\frac{\ell}{2}}(\partial\mathcal{A})$. In terms of this data, we then obtain the following useful formulas.

Lemma 11 (Some elementary identities). *It holds*

$$(108) \quad \nabla P_{\partial\mathcal{A}}(x) = \sum_{i=1}^{d-1} \frac{1}{1 - s_{\partial\mathcal{A}}(x) \bar{\kappa}_i(x)} \bar{\mathbf{t}}_{\partial\mathcal{A}}^i(x) \otimes \bar{\mathbf{t}}_{\partial\mathcal{A}}^i(x).$$

Furthermore, it holds

$$(109) \quad \xi(x) = \bar{\mathbf{n}}_{\partial\mathcal{A}}(x) = \nabla s_{\partial\mathcal{A}}(x), \quad x \in B_{\frac{\ell}{2}}(\partial\mathcal{A}),$$

as well as

$$(110) \quad \nabla \xi(x) = - \sum_{i=1}^{d-1} \frac{\bar{\kappa}_{\partial\mathcal{A}}^i(x)}{1 - s_{\partial\mathcal{A}}(x) \bar{\kappa}_i(x)} \bar{\mathbf{t}}_{\partial\mathcal{A}}^i(x) \otimes \bar{\mathbf{t}}_{\partial\mathcal{A}}^i(x), \quad x \in B_{\frac{\ell}{2}}(\partial\mathcal{A}),$$

and thus

$$(111) \quad \nabla \cdot \xi(x) = -\bar{H}_{\partial\mathcal{A}} - \sum_{i=1}^{d-1} \frac{(\bar{\kappa}_{\partial\mathcal{A}}^i)^2(x)}{1 - s_{\partial\mathcal{A}}(x) \bar{\kappa}_i(x)} s_{\partial\mathcal{A}}(x) \quad x \in B_{\frac{\ell}{2}}(\partial\mathcal{A}).$$

Finally, for all $f \in L^1(B_{\frac{\ell}{2}}(\partial\mathcal{A}); dx)$ it holds

$$(112) \quad \int_{B_{\frac{\ell}{2}}(\partial\mathcal{A})} f dx = \int_{\partial\mathcal{A}} \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} f(y + s\mathbf{n}_{\partial\mathcal{A}}(y)) \prod_{i=1}^{d-1} (1 - s\kappa_{\partial\mathcal{A}}^i(y)) ds d\mathcal{H}^{d-1}.$$

In order to pull off the strategy for the proof of (72) as mentioned before the statement of Proposition 8, we next introduce a C^1 diffeomorphism $\Psi^h: \Omega \rightarrow \Omega$ with the property

$$(113) \quad \Psi^h(A) = \mathcal{A}.$$

This can be done as follows. Let $\bar{\zeta} \in C_{cpt}^\infty(\mathbb{R}; [0, 1])$ such that $\text{supp } \bar{\zeta} \subset [-1/2, 1/2]$, $\bar{\zeta} \equiv 1$ on $[-1/4, 1/4]$ and $|\bar{\zeta}'| \leq 8$. Based on this auxiliary cutoff, define a map

$$(114) \quad \zeta: \mathbb{R}^d \rightarrow [0, 1], \quad x \mapsto \bar{\zeta}\left(\frac{s_{\partial\mathcal{A}}(x)}{\ell}\right).$$

We may then define the desired map Ψ^h by means of

$$(115) \quad \Psi^h(x) := x - \zeta(x)\bar{h}(x)\bar{\mathbf{n}}_{\partial\mathcal{A}}(x).$$

We collect in the upcoming result some basic facts about Ψ^h .

Lemma 12 (Some further useful facts). *We have throughout Ω*

$$(116) \quad \begin{aligned} \nabla\Psi^h &= \text{Id} - (\bar{\mathbf{n}}_{\partial\mathcal{A}} \cdot \nabla\zeta)\bar{h}\bar{\mathbf{n}}_{\partial\mathcal{A}} \otimes \bar{\mathbf{n}}_{\partial\mathcal{A}} \\ &+ \zeta \sum_{i=1}^{d-1} \frac{\bar{\kappa}_{\partial\mathcal{A}}^i}{1 - s_{\partial\mathcal{A}}\bar{\kappa}_{\partial\mathcal{A}}^i} \bar{h}\bar{\mathbf{t}}_{\partial\mathcal{A}}^i \otimes \bar{\mathbf{t}}_{\partial\mathcal{A}}^i, \\ &- \zeta \sum_{i=1}^{d-1} \frac{((\bar{\mathbf{t}}_{\partial\mathcal{A}}^i \cdot \nabla_{\partial\mathcal{A}})h)}{1 - s_{\partial\mathcal{A}}\bar{\kappa}_{\partial\mathcal{A}}^i} \bar{\mathbf{n}}_{\partial\mathcal{A}} \otimes \bar{\mathbf{t}}_{\partial\mathcal{A}}^i, \end{aligned}$$

and thus throughout Ω

$$(117) \quad \det \nabla\Psi^h = (1 - (\bar{\mathbf{n}}_{\partial\mathcal{A}} \cdot \nabla\zeta)\bar{h}) \prod_{i=1}^{d-1} \left(1 + \zeta \frac{\bar{\kappa}_{\partial\mathcal{A}}^i}{1 - s_{\partial\mathcal{A}}\bar{\kappa}_{\partial\mathcal{A}}^i} \bar{h}\right).$$

In particular, $\Psi^h: \Omega \rightarrow \Omega$ is a $C^{1,1-\frac{d-1}{4}}$ diffeomorphism and

$$(118) \quad \Psi^h(x) = x - \bar{h}(x)\bar{\mathbf{n}}_{\partial\mathcal{A}}(x), \quad x \in B_{2\|h\|_{L^\infty(\partial\mathcal{A})}}(\partial\mathcal{A}),$$

$$(119) \quad (\Psi^h)^{-1}(x) = x + \bar{h}(x)\bar{\mathbf{n}}_{\partial\mathcal{A}}(x), \quad x \in B_{2\|h\|_{L^\infty(\partial\mathcal{A})}}(\partial\mathcal{A}).$$

Furthermore, for all $f \in L^1(\partial^*A; d\mathcal{H}^{d-1})$ it holds

$$(120) \quad \begin{aligned} &\int_{\partial^*A} f d\mathcal{H}^{d-1} \\ &= \int_{\partial\mathcal{A}} f \circ (\Psi^h)^{-1} \Big|_{\partial\mathcal{A}} \sqrt{1 + \sum_{i=1}^{d-1} \left(\frac{(\mathbf{t}_{\partial\mathcal{A}}^i \cdot \nabla_{\partial\mathcal{A}})h}{1 - h\kappa_{\partial\mathcal{A}}^i}\right)^2} \prod_{i=1}^{d-1} (1 - h\kappa_{\partial\mathcal{A}}^i) d\mathcal{H}^{d-1}. \end{aligned}$$

Finally,

$$(121) \quad \mathbf{n}_{\partial^*A} \circ (\Psi^h)^{-1} \Big|_{\partial\mathcal{A}} = \frac{1}{\sqrt{1 + \sum_{i=1}^{d-1} \left(\frac{(\mathbf{t}_{\partial\mathcal{A}}^i \cdot \nabla_{\partial\mathcal{A}})h}{1 - h\kappa_{\partial\mathcal{A}}^i}\right)^2}} \left(\mathbf{n}_{\partial\mathcal{A}} - \sum_{i=1}^{d-1} \frac{(\mathbf{t}_{\partial\mathcal{A}}^i \cdot \nabla_{\partial\mathcal{A}})h}{1 - h\kappa_{\partial\mathcal{A}}^i} \mathbf{t}_{\partial\mathcal{A}}^i \right)$$

and for all $\delta > 0$ one may choose $C \gg_\delta 1$ from (69) such that

$$(122) \quad \left| (\text{tr}_{\partial^* A} w) \circ (\Psi^h)^{-1} \Big|_{\partial \mathcal{A}} - \left(H_{\partial \mathcal{A}} + \left(\sum_{i=1}^{d-1} (\kappa_{\partial \mathcal{A}}^i)^2 \right) h + \Delta_{\partial \mathcal{A}} h \right) \right| \leq \delta \max \{ |h|, |\nabla_{\partial \mathcal{A}} h|, |\text{Hess}_{\partial \mathcal{A}} h| \}.$$

The next ingredient is concerned with a representation of the error functionals in terms of the graph function.

Lemma 13 (Error control in perturbative setting). *For any $\delta \in (0, 1)$ one may choose $C \gg_\delta 1$ from (69) such that*

$$(123) \quad (1-\delta) \int_{\partial \mathcal{A}} \frac{1}{2} |\nabla_{\partial \mathcal{A}} h|^2 d\mathcal{H}^{d-1} \leq E_{rel}[A, \mu | \mathcal{A}] \leq (1+\delta) \int_{\partial \mathcal{A}} \frac{1}{2} |\nabla_{\partial \mathcal{A}} h|^2 d\mathcal{H}^{d-1},$$

$$(124) \quad (1-\delta) \int_{\partial \mathcal{A}} \frac{1}{2} \left(\frac{h}{\ell} \right)^2 d\mathcal{H}^{d-1} \leq E_{vol}[A | \mathcal{A}] \leq (1+\delta) \int_{\partial \mathcal{A}} \frac{1}{2} \left(\frac{h}{\ell} \right)^2 d\mathcal{H}^{d-1}$$

and

$$(125) \quad (1-\delta) \mathcal{H}^{d-1}(\partial \mathcal{A}) \leq \mathcal{H}^{d-1}(\partial^* A \cap \Omega) \leq (1+\delta) \mathcal{H}^{d-1}(\partial \mathcal{A}).$$

In the following result, we record the PDEs satisfied by the chemical potentials after pulling back the domain A to \mathcal{A} .

Lemma 14 (PDEs satisfied by transformed chemical potentials). *For arbitrary $v \in H^1(\Omega)$, we define $v_h := v \circ (\Psi^h)^{-1}$. We also define the uniformly elliptic and bounded coefficient field $a^h := \frac{1}{|\det \Psi^h|} (\nabla \Psi^h)^\top \nabla \Psi^h$. Then, $(w - \tilde{w})_h \in H^1(\Omega)$ satisfies*

$$(126) \quad \Delta(w - \tilde{w})_h = \nabla \cdot ((\text{Id} - a^h) \nabla (w - \tilde{w})_h) \quad \text{in } \Omega \setminus \partial \mathcal{A},$$

$$(127) \quad \text{tr}_{\partial \mathcal{A}}(w - \tilde{w})_h = (\text{tr}_{\partial^* A}(w) + \nabla \cdot \xi) \circ (\Psi^h)^{-1} \quad \text{on } \partial \mathcal{A},$$

$$(128) \quad (\mathbf{n}_{\partial \Omega} \cdot \nabla)(w - \tilde{w})_h = 0 \quad \text{on } \partial \Omega.$$

Writing $\bar{u} \in H^1(\Omega)$ for the chemical potential associated with the smoothly evolving phase \mathcal{A} , we also get for $\bar{u} - \tilde{w}_h \in H^1(\Omega)$

$$(129) \quad \Delta(\bar{u} - \tilde{w}_h) = \nabla \cdot ((a^h - \text{Id}) \nabla \tilde{w}_h) \quad \text{in } \Omega \setminus \partial \mathcal{A},$$

$$(130) \quad \text{tr}_{\partial \mathcal{A}}(\bar{u} - \tilde{w}_h) = (H_{\partial \mathcal{A}} + (\nabla \cdot \xi) \circ (\Psi^h)^{-1}) \quad \text{on } \partial \mathcal{A},$$

$$(131) \quad (\mathbf{n}_{\partial \Omega} \cdot \nabla)(\bar{u} - \tilde{w}_h) = 0 \quad \text{on } \partial \Omega.$$

In order to exploit the Hilbert space structure of $H_{MS}^{1/2}(\partial \mathcal{A})$ for an estimate of R_{dissip} and A_{dissip} , respectively, we have to smuggle in the averages of the associated chemical potentials. The following result ensures that this can be done solely at the cost of controlled quantities.

Lemma 15 (Smuggling in averages of chemical potentials). *For any $\delta \in (0, 1)$ one may choose $C \gg_\delta 1$ from (69) and $M \gg_\delta 1$ from (71) such that for some universal constant $\tilde{C} > 0$*

$$(132) \quad \left| \int_{\partial^* A \cap \Omega} w - \tilde{w} d\mathcal{H}^{d-1} \right| + \left| \int_{\partial \mathcal{A}} (w - \tilde{w})_h d\mathcal{H}^{d-1} \right| \leq \delta C_{PS}(\mathcal{A}) \left(\int_{\Omega} |\nabla(w - \tilde{w})|^2 dx \right)^{\frac{1}{2}} + \frac{\tilde{C}}{\ell} \frac{\sqrt{E[A | \mathcal{A}]}}{\sqrt{\mathcal{H}^{d-1}(\partial^* A \cap \Omega)}},$$

where $C_{PS}(\mathcal{A})$ is the constant from the Sobolev–Poincaré trace inequality: for any $v \in H^1(\mathcal{A})$,

$$(133) \quad \left(\int_{\partial\mathcal{A}} \left(v - \int_{\partial\mathcal{A}} v d\mathcal{H}^{d-1} \right)^2 d\mathcal{H}^{d-1} \right)^{\frac{1}{2}} \leq C_{PS}(\mathcal{A}) \left(\int_{\mathcal{A}} |\nabla v|^2 dx \right)^{\frac{1}{2}}.$$

Furthermore,

$$(134) \quad \left| \int_{\partial^* A \cap \Omega} n_{\partial^* A} \cdot (B - \llbracket \nabla \tilde{w} \rrbracket) d\mathcal{H}^{d-1} \right| \leq \tilde{C} \|n_{\partial\mathcal{A}} \cdot \llbracket \nabla \tilde{u} \rrbracket\|_{L^\infty(\partial\mathcal{A})} \sqrt{\mathcal{H}^{d-1}(\partial\mathcal{A})} \sqrt{E_{vol}[A|\mathcal{A}]}. \quad (134)$$

We also rely on a regularity estimate for the transformed chemical potential \tilde{w}_h .

Lemma 16 (Schauder estimate for transformed chemical potential). *There exists a constant $C_{reg}(\mathcal{A}, \Lambda) \in (0, \infty)$ such that*

$$(135) \quad \|\tilde{w}_h\|_{C^{1, \frac{1}{2}}(\overline{\mathcal{A} \cap B_{\ell/2}(\partial\mathcal{A})})} + \|\tilde{w}_h\|_{C^{1, \frac{1}{2}}(\overline{(\Omega \setminus \mathcal{A}) \cap B_{\ell/2}(\partial\mathcal{A})})} \leq C_{reg}(\mathcal{A}, \Lambda).$$

An analogous estimate holds for $(\tilde{w}_\vartheta)_h$.

Another useful input is given by energy estimates.

Lemma 17 (Energy estimates for transformed chemical potentials). *Decompose $(w - \tilde{w})_h = v_h^{(1)} + v_h^{(2)}$, where $v_h^{(1)} \in H^1(\Omega)$ denotes the unique solution of*

$$(136) \quad \Delta v_h^{(1)} = 0 \quad \text{in } \Omega \setminus \partial\mathcal{A},$$

$$(137) \quad \text{tr}_{\partial\mathcal{A}} v_h^{(1)} = (\text{tr}_{\partial^* A}(w) + \nabla \cdot \xi) \circ (\Psi^h)^{-1} \quad \text{on } \partial\mathcal{A},$$

$$(138) \quad (n_{\partial\Omega} \cdot \nabla) v_h^{(1)} = 0 \quad \text{on } \partial\Omega.$$

For any $\delta \in (0, 1)$ one may choose $C \gg_\delta 1$ from (69) such that

$$(139) \quad \int_{\Omega} |\nabla v_h^{(1)}|^2 dx \leq (1+\delta) \int_{\Omega} |\nabla(w - \tilde{w})_h|^2 dx,$$

$$(140) \quad \int_{\Omega} |\nabla(w - \tilde{w})_h|^2 dx \leq (1+\delta) \int_{\Omega} |\nabla(w - \tilde{w})|^2 dx.$$

There also exists a universal constant $\tilde{C} > 0$ such that

$$(141) \quad \left| \int_{\Omega} \nabla v_h^{(1)} \cdot (\text{Id} - a^h) \nabla \tilde{w}_h dx \right| \leq \tilde{C} \left(\int_{\Omega} |\nabla v_h^{(1)}|^2 dx \right)^{\frac{1}{2}} C_{reg}(\mathcal{A}, \Lambda) \sqrt{E[A|\mathcal{A}]},$$

where $C_{reg}(\mathcal{A}, \Lambda)$ is the constant from Lemma 16.

The final ingredient consists of an interpolation estimate, transferring control in terms of the Hilbert space structure on $H_{MS}^{1/2}(\partial\mathcal{A})$ to control in terms of the standard H^1 and H^2 norm on $\partial\mathcal{A}$ (and therefore, for our purposes, to control in terms of our error functional and the dissipation).

Lemma 18 (Interpolation estimate). *There exists a constant $C_{int}(\mathcal{A}, \Omega) \in (0, \infty)$ such that for all $f \in H^2(\partial\mathcal{A})$ with $\int_{\partial\mathcal{A}} f d\mathcal{H}^{d-1} = 0$ it holds*

$$(142) \quad \|f\|_{H_{MS}^{1/2}(\partial\mathcal{A})} \leq C_{int}(\mathcal{A}, \Omega) \|f\|_{H^1(\partial\mathcal{A})}^{\frac{1}{2}} \|f\|_{H^2(\partial\mathcal{A})}^{\frac{1}{2}}.$$

We have now everything in place to proceed with the proofs.

Proof of Proposition 8, Part I: Estimate (72). Next to the decomposition $(w-\tilde{w})_h = v_h^{(1)} + v_h^{(2)}$ as defined in Lemma 17, decompose also $\bar{u}-\tilde{w}_h = \bar{v}_h^{(1)} + \bar{v}_h^{(2)}$, where $\bar{v}_h^{(1)} \in H^1(\Omega)$ denotes the unique solution of

$$(143) \quad \Delta \bar{v}_h^{(1)} = 0 \quad \text{in } \Omega \setminus \partial\mathcal{A},$$

$$(144) \quad \text{tr}_{\partial\mathcal{A}} \bar{v}_h^{(1)} = (H_{\partial\mathcal{A}} + (\nabla \cdot \xi) \circ (\Psi^h)^{-1}) \quad \text{on } \partial\mathcal{A},$$

$$(145) \quad (\mathbf{n}_{\partial\Omega} \cdot \nabla) \bar{v}_h^{(1)} = 0 \quad \text{on } \partial\Omega.$$

The main claim of the proof then is that the argument for the estimate (72) boils down to an estimate of

$$(146) \quad \text{Err} := \left| \int_{\partial\mathcal{A}} \left(v_h^{(1)} - \int_{\partial\mathcal{A}} v_h^{(1)} d\mathcal{H}^{d-1} \right) \mathbf{n}_{\partial\mathcal{A}} \cdot \llbracket \nabla \bar{v}_h^{(1)} \rrbracket d\mathcal{H}^{d-1} \right|.$$

The estimate for *Err* itself will be a consequence of the Hilbert space structure defined on $H_{MS}^{1/2}(\partial\mathcal{A})$, the interpolation estimate (142), that our error functional controls the H^1 norm on $\partial\mathcal{A}$ of the height function h , see (123) and (124), and that the dissipation term $\int_{\Omega} \frac{1}{2} |(\nabla w - \nabla \tilde{w})(\cdot, t)|^2 dx$ controls second-order derivatives of h .

Step 1: Reduction argument. Recall that we want to estimate

$$(147) \quad \widetilde{\text{Err}} := \left| \int_{\partial^* A \cap \Omega} (w-\tilde{w}) \mathbf{n}_{\partial^* A} \cdot (B + \llbracket \nabla \tilde{w} \rrbracket) d\mathcal{H}^{d-1} \right|.$$

To this end, we first smuggle in the average $\int_{\partial\mathcal{A}} (w-\tilde{w})_h d\mathcal{H}^{d-1}$ allowing us to deduce from (132), (134) and (125) that

$$(148) \quad \begin{aligned} \widetilde{\text{Err}} &\leq \left| \int_{\partial^* A \cap \Omega} \left((w-\tilde{w}) - \int_{\partial\mathcal{A}} (w-\tilde{w})_h d\mathcal{H}^{d-1} \right) \mathbf{n}_{\partial^* A} \cdot (B + \llbracket \nabla \tilde{w} \rrbracket) d\mathcal{H}^{d-1} \right| \\ &\quad + \frac{\delta}{8} \int_{\Omega} |\nabla(w-\tilde{w})|^2 dx + \frac{\tilde{C}}{\delta} E[A, \mu|_{\mathcal{A}}]. \end{aligned}$$

In a second step, we will transform by an application of the area formula (120) the integral on $\partial^* A \cap \Omega$ into an integral on $\partial\mathcal{A}$. Note to this end that, thanks to the $C^{1, \frac{1}{2}}$ -regularity of \tilde{w}_h established in Lemma 16, the jump of $(\nabla \tilde{w})_h$ across $\partial\mathcal{A}$ in direction of $\mathbf{n}_{\partial^* A} \circ (\Psi^h)^{-1}|_{\partial\mathcal{A}}$ is equal to the jump of $(\nabla \tilde{w})_h$ across $\partial\mathcal{A}$ in direction of $\mathbf{n}_{\partial\mathcal{A}}$. Hence, recalling also from (26) that $B = (\bar{\mathbf{n}}_{\partial\mathcal{A}} \cdot (\llbracket \nabla \bar{u} \rrbracket \circ P_{\partial\mathcal{A}})) \bar{\mathbf{n}}_{\partial\mathcal{A}}$, we obtain by an application of (120) as well as (121) that

$$(149) \quad \begin{aligned} &\left| \int_{\partial^* A \cap \Omega} \left((w-\tilde{w}) - \int_{\partial\mathcal{A}} (w-\tilde{w})_h d\mathcal{H}^{d-1} \right) \mathbf{n}_{\partial^* A} \cdot (B + \llbracket \nabla \tilde{w} \rrbracket) d\mathcal{H}^{d-1} \right| \\ &\leq \left| \int_{\partial\mathcal{A}} \prod_{i=1}^{d-1} (1-h\kappa_{\partial\mathcal{A}}^i) \left((w-\tilde{w})_h - \int_{\partial\mathcal{A}} (w-\tilde{w})_h d\mathcal{H}^{d-1} \right) \mathbf{n}_{\partial\mathcal{A}} \cdot \llbracket \nabla \bar{u} - (\nabla \tilde{w})_h \rrbracket d\mathcal{H}^{d-1} \right| \\ &\quad + \sum_{i=1}^{d-1} \left| \int_{\partial\mathcal{A}} \prod_{j=1, j \neq i}^{d-1} (1-h\kappa_{\partial\mathcal{A}}^j) (\mathbf{t}_{\partial\mathcal{A}}^i \cdot \nabla_{\partial\mathcal{A}}) h \right. \\ &\quad \quad \left. \times \left((w-\tilde{w})_h - \int_{\partial\mathcal{A}} (w-\tilde{w})_h d\mathcal{H}^{d-1} \right) \mathbf{t}_{\partial\mathcal{A}}^i \cdot \llbracket \nabla \bar{u} - (\nabla \tilde{w})_h \rrbracket d\mathcal{H}^{d-1} \right| \\ &=: \widetilde{\text{Err}}_1 + \widetilde{\text{Err}}_2. \end{aligned}$$

We proceed with an estimate for \widetilde{Err}_2 . To this end, we first record the following elementary identities:

$$(150) \quad \mathbf{n}_{\partial\mathcal{A}} \cdot \llbracket (\nabla \tilde{w})_h \rrbracket = \mathbf{n}_{\partial\mathcal{A}} \cdot \llbracket \nabla(\tilde{w}_h) \rrbracket \quad \text{on } \partial\mathcal{A},$$

$$(151) \quad \mathbf{t}_{\partial\mathcal{A}}^i \cdot \llbracket (\nabla \tilde{w})_h \rrbracket = \left(\frac{1}{1-h\kappa_{\partial\mathcal{A}}^i} \mathbf{t}_{\partial\mathcal{A}}^i - \frac{(\mathbf{t}_{\partial\mathcal{A}}^i \cdot \nabla_{\partial\mathcal{A}})h}{1-h\kappa_{\partial\mathcal{A}}^i} \mathbf{n}_{\partial\mathcal{A}} \right) \cdot \llbracket \nabla(\tilde{w}_h) \rrbracket \quad \text{on } \partial\mathcal{A}.$$

Indeed, the claims (150)–(151) are direct consequences of the chain rule in the form of $(\nabla \tilde{w})_h = \nabla(\tilde{w}_h)(\nabla \Psi^h \circ (\Psi^h)^{-1})$ and the formulas (116) and (119). Hence, we obtain from an application of the Cauchy–Schwarz inequality, Young’s inequality, the Sobolev–Poincaré trace inequality (133), and the estimates (140), (135) and (123)

$$(152) \quad \widetilde{Err}_2 \leq \frac{\delta}{8} \int_{\Omega} |\nabla(w-\tilde{w})|^2 dx + \frac{\tilde{C}}{\delta} E[A, \mu|_{\mathcal{A}}].$$

It remains to control \widetilde{Err}_1 . Recalling that $|\kappa_{\partial\mathcal{A}}^i| \leq 1/\ell$, it follows analogously to (152), relying this time on (124) instead of (123) and (150) instead of (151), that

$$(153) \quad \begin{aligned} \widetilde{Err}_1 \leq & \left| \int_{\partial\mathcal{A}} \left((w-\tilde{w})_h - \int_{\partial\mathcal{A}} (w-\tilde{w})_h d\mathcal{H}^{d-1} \right) \mathbf{n}_{\partial\mathcal{A}} \cdot \llbracket \nabla(\bar{u}-\tilde{w}_h) \rrbracket d\mathcal{H}^{d-1} \right| \\ & + \frac{\delta}{8} \int_{\Omega} |\nabla(w-\tilde{w})|^2 dx + \frac{\tilde{C}}{\delta} E[A, \mu|_{\mathcal{A}}]. \end{aligned}$$

Recalling the decompositions $(w-\tilde{w})_h = v_h^{(1)} + v_h^{(2)}$ and $\bar{u}-\tilde{w}_h = \bar{v}_h^{(1)} + \bar{v}_h^{(2)}$, where we in addition note that $v_h^{(2)} = \bar{v}_h^{(2)} = 0$ on $\partial\mathcal{A}$ by virtue of the definitions of $v_h^{(1)}$ and $\bar{v}_h^{(1)}$, we further estimate by triangle inequality and the definition (146) of Err

$$(154) \quad \begin{aligned} & \left| \int_{\partial\mathcal{A}} \left((w-\tilde{w})_h - \int_{\partial\mathcal{A}} (w-\tilde{w})_h d\mathcal{H}^{d-1} \right) \mathbf{n}_{\partial\mathcal{A}} \cdot \llbracket \nabla(\bar{u}-\tilde{w}_h) \rrbracket d\mathcal{H}^{d-1} \right| \\ & \leq Err + \left| \int_{\partial\mathcal{A}} \left(v_h^{(1)} - \int_{\partial\mathcal{A}} v_h^{(1)} d\mathcal{H}^{d-1} \right) \mathbf{n}_{\partial\mathcal{A}} \cdot \llbracket \nabla \bar{v}_h^{(2)} \rrbracket d\mathcal{H}^{d-1} \right| \\ & =: Err + \widetilde{Err}'_1. \end{aligned}$$

In order to estimate the remaining term \widetilde{Err}'_1 , we record that

$$(155) \quad \Delta \bar{v}_h^{(2)} = \nabla \cdot ((a^h - \text{Id}) \nabla \tilde{w}_h) \quad \text{in } \Omega \setminus \partial\mathcal{A},$$

$$(156) \quad \text{tr}_{\partial\mathcal{A}} \bar{v}_h^{(2)} = 0 \quad \text{on } \partial\mathcal{A},$$

$$(157) \quad (\mathbf{n}_{\partial\Omega} \cdot \nabla) \bar{v}_h^{(2)} = 0 \quad \text{on } \partial\Omega.$$

Hence, testing in a first step the PDE satisfied by $\bar{v}_h^{(2)}$ with the test function $v_h^{(1)} - \int_{\partial\mathcal{A}} v_h^{(1)} d\mathcal{H}^{d-1}$, and in a second step the PDE satisfied by $v_h^{(1)}$ with the test function $\bar{v}_h^{(2)}$, we get

$$\begin{aligned} & \int_{\partial\mathcal{A}} \left(v_h^{(1)} - \int_{\partial\mathcal{A}} v_h^{(1)} d\mathcal{H}^{d-1} \right) \mathbf{n}_{\partial\mathcal{A}} \cdot \llbracket \nabla \bar{v}_h^{(2)} \rrbracket d\mathcal{H}^{d-1} \\ & = \int_{\Omega} \nabla v_h^{(1)} \cdot \nabla \bar{v}_h^{(2)} dx - \int_{\Omega} \nabla v_h^{(1)} \cdot ((a^h - \text{Id}) \nabla \tilde{w}_h) dx \\ & \quad + \int_{\partial\mathcal{A}} \left(v_h^{(1)} - \int_{\partial\mathcal{A}} v_h^{(1)} d\mathcal{H}^{d-1} \right) \mathbf{n}_{\partial\mathcal{A}} \cdot (a^h - \text{Id}) \llbracket \nabla \tilde{w}_h \rrbracket d\mathcal{H}^{d-1} \end{aligned}$$

$$(158) \quad = - \int_{\Omega} \nabla v_h^{(1)} \cdot ((a^h - \text{Id}) \nabla \tilde{w}_h) dx \\ + \int_{\partial \mathcal{A}} \left((w - \tilde{w})_h - \int_{\partial \mathcal{A}} (w - \tilde{w})_h d\mathcal{H}^{d-1} \right) \mathbf{n}_{\partial \mathcal{A}} \cdot (a^h - \text{Id}) [\nabla \tilde{w}_h] d\mathcal{H}^{d-1}.$$

In particular, based on the same arguments leading to, e.g., (152), the definition of a^h , the formulas (116) and (117), as well as the estimates from Lemma 17, we deduce from (158) that

$$(159) \quad \widetilde{Err}'_1 \leq \frac{\delta}{8} \int_{\Omega} |\nabla(w - \tilde{w})|^2 dx + \frac{\tilde{C}}{\delta} E[A, \mu | \mathcal{A}].$$

In summary, comparing our estimates (148)–(154) and (159) with the claim (72), we see that we indeed reduced matters to an estimate of (146).

Step 2: Estimate for (146). Note that (111) and (119) directly entail

$$H_{\partial \mathcal{A}} + (\nabla \cdot \xi) \circ (\Psi^h)^{-1} = - \sum_{i=1}^{d-1} \frac{(\kappa_{\partial \mathcal{A}}^i)^2}{1 - h \kappa_{\partial \mathcal{A}}^i} h =: f_h \quad \text{on } \partial \mathcal{A}.$$

Furthermore, the very definition of the Hilbert space structure on $H_{MS}^{1/2}(\partial \mathcal{A})$ yields

$$Err = \left| \left\langle v_h^{(1)} - \int_{\partial \mathcal{A}} v_h^{(1)} d\mathcal{H}^{d-1}, f_h - \int_{\partial \mathcal{A}} f_h d\mathcal{H}^{d-1} \right\rangle_{H_{MS}^{1/2}(\partial \mathcal{A})} \right|.$$

Hence, from the Cauchy–Schwarz inequality, the definition of the norm on $H_{MS}^{1/2}(\partial \mathcal{A})$, i.e. $\|v_h^{(1)} - \int_{\partial \mathcal{A}} v_h^{(1)} d\mathcal{H}^{d-1}\|_{H_{MS}^{1/2}(\partial \mathcal{A})}^2 = \int_{\Omega} |\nabla v_h^{(1)}|^2 dx$, the estimates (139)–(140) we infer

$$(160) \quad Err \leq \frac{\delta}{16} \int_{\Omega} |\nabla(w - \tilde{w})|^2 dx + \frac{\tilde{C}}{\delta} \left\| f_h - \int_{\partial \mathcal{A}} f_h \right\|_{H_{MS}^{1/2}(\partial \mathcal{A})}^2.$$

Denoting by $L_{\partial \mathcal{A}}$ the Weingarten tensor of $\partial \mathcal{A}$ (see, e.g., [25, Equation (2.7), page 47]), we note that in invariant notation $f_h = \text{trace}(L_{\partial \mathcal{A}}^2 (\text{Id} - h L_{\partial \mathcal{A}})^{-1}) h$, so that together with the interpolation estimate (142) and the estimates (123)–(124) we get as an upgrade of (160)

$$(161) \quad Err \leq \frac{\delta}{16} \int_{\Omega} |\nabla(w - \tilde{w})|^2 dx + \frac{\tilde{C}}{\delta} E[A, \mu | \mathcal{A}] + \frac{\tilde{C}}{\delta} \sqrt{E[A, \mu | \mathcal{A}]} \|\text{Hess}_{\partial \mathcal{A}} h\|_{L^2(\partial \mathcal{A})}.$$

Furthermore, by L^2 Caldéron–Zygmund theory ($\partial \mathcal{A}$ is closed)

$$(162) \quad \|\text{Hess}_{\partial \mathcal{A}} h\|_{L^2(\partial \mathcal{A})} \leq \tilde{C}' \|\Delta_{\partial \mathcal{A}} h\|_{L^2(\partial \mathcal{A})}.$$

Since by (111), (122), (123)–(124) and (126)

$$(163) \quad \|\Delta_{\partial \mathcal{A}} h\|_{L^2(\partial \mathcal{A})} \leq \tilde{C}''' (\|(w - \tilde{w})_h\|_{L^2(\partial \mathcal{A})} + \sqrt{E[A, \mu | \mathcal{A}]}) + \delta' \tilde{C}'' \|\text{Hess}_{\partial \mathcal{A}} h\|_{L^2(\partial \mathcal{A})},$$

where $\delta' \in (0, 1)$ will be specified in a moment, and since by (132), (133) and (140)

$$(164) \quad \|(w - \tilde{w})_h\|_{L^2(\partial \mathcal{A})} \leq \tilde{C}'''' (\|\nabla(w - \tilde{w})\|_{L^2(\Omega)} + \sqrt{E[A, \mu | \mathcal{A}]}) ,$$

it follows from choosing $\delta' \ll \tilde{C}', \tilde{C}''$ and the previous four displays that

$$(165) \quad Err \leq \frac{\delta}{8} \int_{\Omega} |\nabla(w - \tilde{w})|^2 dx + \frac{\tilde{C}''''}{\delta} E[A, \mu | \mathcal{A}].$$

Together with the estimates from Step 1, this concludes the proof of (72). \square

Proof of Lemma 11. The identity (109) simply follows from (13) and $|\nabla s_{\partial\mathcal{A}}| = 1$. The identities (110) and (111) in turn follow from (108),

$$(166) \quad \nabla_{\partial\mathcal{A}} \mathbf{n}_{\partial\mathcal{A}} = - \sum_{i=1}^{d-1} \kappa_i \mathbf{t}_{\partial\mathcal{A}}^i \otimes \mathbf{t}_{\partial\mathcal{A}}^i,$$

and the chain rule in the form of $\nabla\xi = (\nabla_{\partial\mathcal{A}} \mathbf{n}_{\partial\mathcal{A}} \circ P_{\partial\mathcal{A}}) \nabla P_{\partial\mathcal{A}}$. Since (112) is simply a consequence of the coarea formula applied to the tubular neighborhood diffeomorphism

$$(167) \quad \Phi: B_{\ell/2}(\partial\mathcal{A}) \rightarrow \partial\mathcal{A} \times (-\ell/2, \ell/2), \quad x \mapsto (P_{\partial\mathcal{A}}(x), s_{\partial\mathcal{A}}(x)),$$

it only remains to prove (108).

To this end, we first note that

$$(168) \quad P_{\partial\mathcal{A}}(x) = x - s_{\partial\mathcal{A}}(x) \bar{\mathbf{n}}_{\partial\mathcal{A}}(x), \quad x \in B_{\frac{\ell}{2}}(\partial\mathcal{A}),$$

and therefore

$$(169) \quad \nabla P_{\partial\mathcal{A}} = (\text{Id} - \bar{\mathbf{n}}_{\partial\mathcal{A}} \otimes \bar{\mathbf{n}}_{\partial\mathcal{A}}) - s_{\partial\mathcal{A}} \nabla \bar{\mathbf{n}}_{\partial\mathcal{A}} \quad \text{in } B_{\frac{\ell}{2}}(\partial\mathcal{A}).$$

In particular, $\nabla P_{\partial\mathcal{A}}$ is symmetric (recall that $\nabla \bar{\mathbf{n}}_{\partial\mathcal{A}} = \text{Hess} s_{\partial\mathcal{A}}$), so that by an application of the chain rule (again in the form of $\nabla \bar{\mathbf{n}}_{\partial\mathcal{A}} = (\nabla_{\partial\mathcal{A}} \mathbf{n}_{\partial\mathcal{A}} \circ P_{\partial\mathcal{A}}) \nabla P_{\partial\mathcal{A}}$) it follows that $(\bar{\mathbf{t}}_{\partial\mathcal{A}}^i \cdot \nabla) P_{\partial\mathcal{A}} = \frac{1}{1 - s_{\partial\mathcal{A}} \bar{\kappa}_i} \bar{\mathbf{t}}_{\partial\mathcal{A}}^i$. Since furthermore $(\nabla P_{\partial\mathcal{A}})^\top \bar{\mathbf{n}}_{\partial\mathcal{A}} = (\bar{\mathbf{n}}_{\partial\mathcal{A}} \cdot \nabla) P_{\partial\mathcal{A}} = 0$, we indeed obtain (108). \square

Proof of Lemma 12. Thanks to Lemma 11, the formulas (116)–(117) are just a straightforward computation starting from the definition (115). The formulas (118) and (119) in turn directly follow from $\text{supp } \bar{\zeta} \subset [-1/2, 1/2]$, the definition (114), and (69). Furthermore, (120) is just the area formula applied to the map

$$(170) \quad (\Psi^h)^{-1}|_{\partial\mathcal{A}}: \partial\mathcal{A} \rightarrow \partial^* A.$$

Next, we claim that (in the sense of weak derivatives)

$$(171) \quad \text{tr}_{\partial^* A}(w) \mathbf{n}_{\partial^* A} = \nabla_{\partial^* A} \cdot (\text{Id} - \mathbf{n}_{\partial^* A} \otimes \mathbf{n}_{\partial^* A}).$$

Indeed, this is true thanks to the Gibbs–Thomson law (7e). Since the right hand side of (171) is precisely the mean curvature vector of $\partial^* A$, the claims (121) and (122) can therefore be directly inferred from [25, Section 2.2]. \square

Proof of Lemma 13. The estimates (125) simply follow from the area formula (120) and (69). For the proof of (123), note that (109) and (121) imply

$$(172) \quad (\mathbf{n}_{\partial^* A} \cdot \xi) \cdot (\Psi^h)^{-1} = \frac{1}{\sqrt{1 + \sum_{i=1}^{d-1} \left(\frac{(\mathbf{t}_{\partial\mathcal{A}}^i \cdot \nabla_{\partial\mathcal{A}}) h}{1 - h \kappa_{\partial\mathcal{A}}^i} \right)^2}},$$

which in turn entails (123) due to (120) and (69). Finally, by the coarea formula (112) as well as the assumptions (14) and (69)

$$(173) \quad E_{\text{vol}}[A|\mathcal{A}] = \int_{\partial\mathcal{A}} \int_0^h \frac{s}{\ell^2} \prod_{i=1}^{d-1} (1 - s \kappa_{\partial\mathcal{A}}^i) ds d\mathcal{H}^{d-1},$$

so that (124) is now a direct consequence of (69). \square

Proof of Lemma 14. These assertions are immediate consequences of the well-known transformation formulas for PDEs in distributional form. \square

Proof of Lemma 15. We proceed in three steps.

Step 1: Proof of (132), Part I. Adding zero, recalling (51), and exploiting the Gibbs–Thomson law (7e) with admissible test function ξ , we obtain

$$\begin{aligned}
\int_{\partial^* A \cap \Omega} w - \tilde{w} \, d\mathcal{H}^{d-1} &= \int_{\partial^* A \cap \Omega} w + \nabla \cdot \xi \, d\mathcal{H}^{d-1} \\
&= \int_{\partial^* A \cap \Omega} w n_{\partial^* A} \cdot \xi \, d\mathcal{H}^{d-1} + \int_{\partial^* A \cap \Omega} w(1 - n_{\partial^* A} \cdot \xi) \, d\mathcal{H}^{d-1} \\
&\quad + \int_{\partial^* A \cap \Omega} (\text{Id} - n_{\partial^* A} \otimes n_{\partial^* A}) : \nabla \xi \, d\mathcal{H}^{d-1} \\
&\quad + \int_{\partial^* A \cap \Omega} n_{\partial^* A} \otimes n_{\partial^* A} : \nabla \xi \, d\mathcal{H}^{d-1} \\
(174) \qquad &= \int_{\partial^* A \cap \Omega} w(1 - n_{\partial^* A} \cdot \xi) \, d\mathcal{H}^{d-1} \\
&\quad + \int_{\partial^* A \cap \Omega} n_{\partial^* A} \otimes n_{\partial^* A} : \nabla \xi \, d\mathcal{H}^{d-1}.
\end{aligned}$$

Based on (109) and (110), we also deduce that along $\partial^* A$

$$\begin{aligned}
n_{\partial^* A} \otimes n_{\partial^* A} : \nabla \xi &= - \sum_{i=1}^{d-1} \frac{\bar{\kappa}_{\partial \mathcal{A}}^i}{1 - h \bar{\kappa}_{\partial \mathcal{A}}^i} (n_{\partial^* A} \cdot \bar{t}_{\partial \mathcal{A}}^i)^2 \\
(175) \qquad &= (\nabla \cdot \xi)(1 - n_{\partial^* A} \cdot \xi) \\
&\quad + (\nabla \cdot \xi) n_{\partial^* A} \cdot \xi (1 - n_{\partial^* A} \cdot \xi) \\
&\quad + \sum_{i=1}^{d-1} \frac{\bar{\kappa}_{\partial \mathcal{A}}^i}{1 - h \bar{\kappa}_{\partial \mathcal{A}}^i} \sum_{j=1, j \neq i}^{d-1} (n_{\partial^* A} \cdot \bar{t}_{\partial \mathcal{A}}^j)^2.
\end{aligned}$$

Hence, inserting (175) back into (174) and adding two times another zero yields (recalling in the process also (51) and (111))

$$\begin{aligned}
\int_{\partial^* A \cap \Omega} w - \tilde{w} \, d\mathcal{H}^{d-1} &= \int_{\partial^* A \cap \Omega} (w - \tilde{w})(1 - n_{\partial^* A} \cdot \xi) \, d\mathcal{H}^{d-1} \\
&\quad + \int_{\partial^* A \cap \Omega} (\nabla \cdot \xi) n_{\partial^* A} \cdot \xi (1 - n_{\partial^* A} \cdot \xi) \, d\mathcal{H}^{d-1} \\
&\quad + \int_{\partial^* A \cap \Omega} \sum_{i=1}^{d-1} \frac{\bar{\kappa}_{\partial \mathcal{A}}^i}{1 - h \bar{\kappa}_{\partial \mathcal{A}}^i} \sum_{j=1, j \neq i}^{d-1} (n_{\partial^* A} \cdot \bar{t}_{\partial \mathcal{A}}^j)^2 \, d\mathcal{H}^{d-1} \\
&= \left(\int_{\partial^* A \cap \Omega} w - \tilde{w} \, d\mathcal{H}^{d-1} \right) E_{rel}[A, \mu | \mathcal{A}] \\
&\quad + \int_{\partial^* A \cap \Omega} \left((w - \tilde{w}) - \int_{\partial^* A \cap \Omega} (w - \tilde{w}) \, d\mathcal{H}^{d-1} \right) (1 - n_{\partial^* A} \cdot \xi) \, d\mathcal{H}^{d-1} \\
&\quad + \int_{\partial^* A \cap \Omega} (\nabla \cdot \xi) n_{\partial^* A} \cdot \xi (1 - n_{\partial^* A} \cdot \xi) \, d\mathcal{H}^{d-1} \\
&\quad + \int_{\partial^* A \cap \Omega} \sum_{i=1}^{d-1} \frac{\bar{\kappa}_{\partial \mathcal{A}}^i}{1 - h \bar{\kappa}_{\partial \mathcal{A}}^i} \sum_{j=1, j \neq i}^{d-1} (n_{\partial^* A} \cdot \bar{t}_{\partial \mathcal{A}}^j)^2 \, d\mathcal{H}^{d-1}
\end{aligned}$$

$$(176) \quad =: I + II + III + IV.$$

By virtue of (125), (71) and $\sum_{j=1, j \neq i}^{d-1} (n_{\partial^* A} \cdot \bar{v}_{\partial \mathcal{A}}^j)^2 \leq 2(1 - n_{\partial^* A} \cdot \xi)$ we then obtain for $M \gg 1$

$$(177) \quad |I + III| \leq \frac{1}{4} \left| \int_{\partial^* A \cap \Omega} w - \tilde{w} d\mathcal{H}^{d-1} \right| + \frac{\tilde{C}}{\ell} E_{rel}[A, \mu | \mathcal{A}],$$

so that we obtain the following upgrade of (176)

$$(178) \quad \left| \int_{\partial^* A \cap \Omega} w - \tilde{w} d\mathcal{H}^{d-1} \right| \leq \tilde{C} |II| + \frac{\tilde{C}}{\ell} E_{rel}[A, \mu | \mathcal{A}].$$

It remains to bound the contribution from II , cf. (176). Because of $(1 - n_{\partial^* A} \cdot \xi)^2 \leq 4(1 - n_{\partial^* A} \cdot \xi)$ we get from an application of Cauchy–Schwarz inequality

$$|II| \leq 2 \left(\int_{\partial^* A \cap \Omega} \left| (w - \tilde{w}) - \fint_{\partial^* A \cap \Omega} (w - \tilde{w}) d\mathcal{H}^{d-1} \right|^2 d\mathcal{H}^{d-1} \right)^{\frac{1}{2}} E_{rel}^{\frac{1}{2}}[A, \mu | \mathcal{A}],$$

so that by adding zero we obtain

$$\begin{aligned} & |II| \\ & \leq 2 \left(\int_{\partial^* A \cap \Omega} \left| (w - \tilde{w}) - \fint_{\partial \mathcal{A}} (w - \tilde{w})_h d\mathcal{H}^{d-1} \right|^2 d\mathcal{H}^{d-1} \right)^{\frac{1}{2}} E_{rel}^{\frac{1}{2}}[A, \mu | \mathcal{A}] \\ & \quad + 2 \sqrt{\mathcal{H}^{d-1}(\partial^* A \cap \Omega)} \left| \fint_{\partial^* A \cap \Omega} (w - \tilde{w}) d\mathcal{H}^{d-1} - \fint_{\partial \mathcal{A}} (w - \tilde{w})_h d\mathcal{H}^{d-1} \right| E_{rel}^{\frac{1}{2}}[A, \mu | \mathcal{A}] \\ & \leq 2 \left(\int_{\partial^* A \cap \Omega} \left| (w - \tilde{w}) - \fint_{\partial \mathcal{A}} (w - \tilde{w})_h d\mathcal{H}^{d-1} \right|^2 d\mathcal{H}^{d-1} \right)^{\frac{1}{2}} E_{rel}^{\frac{1}{2}}[A, \mu | \mathcal{A}] \\ & \quad + 2 \frac{\sqrt{\mathcal{H}^{d-1}(\partial^* A \cap \Omega)}}{\mathcal{H}^{d-1}(\partial \mathcal{A})} \left| \int_{\partial^* A \cap \Omega} (w - \tilde{w}) d\mathcal{H}^{d-1} - \int_{\partial \mathcal{A}} (w - \tilde{w})_h d\mathcal{H}^{d-1} \right| E_{rel}^{\frac{1}{2}}[A, \mu | \mathcal{A}] \\ & \quad + 2 \sqrt{\mathcal{H}^{d-1}(\partial^* A \cap \Omega)} \left| \frac{1}{\mathcal{H}^{d-1}(\partial^* A \cap \Omega)} - \frac{1}{\mathcal{H}^{d-1}(\partial \mathcal{A})} \right| E_{rel}^{\frac{1}{2}}[A, \mu | \mathcal{A}] \left| \int_{\partial^* A \cap \Omega} (w - \tilde{w}) d\mathcal{H}^{d-1} \right| \\ (179) \quad & =: II' + II'' + II'''. \end{aligned}$$

We estimate term by term. For the first term, we simply make use of the area formula (120) (abbreviating in the following the associated area factor by \mathcal{J}_h^{d-1}), the Sobolev–Poincaré trace inequality (133) as well as the estimate (140) to deduce

$$\begin{aligned} II' & \leq \tilde{C} \left(\int_{\partial \mathcal{A}} \left| (w - \tilde{w})_h - \fint_{\partial \mathcal{A}} (w - \tilde{w})_h d\mathcal{H}^{d-1} \right|^2 d\mathcal{H}^{d-1} \right)^{\frac{1}{2}} E_{rel}^{\frac{1}{2}}[A, \mu | \mathcal{A}] \\ (180) \quad & \leq \tilde{C} \sqrt{\mathcal{H}^{d-1}(\partial \mathcal{A})} C_{PS}(\mathcal{A}) E_{rel}^{\frac{1}{2}}[A, \mu | \mathcal{A}] \left(\int_{\Omega} |\nabla(w - \tilde{w})|^2 dx \right)^{\frac{1}{2}}. \end{aligned}$$

For an estimate of the second term, we first recognize through an application of the area formula (120) (abbreviating in the following the associated area factor by \mathcal{J}_h^{d-1}) and adding zero that

$$\left| \int_{\partial^* A \cap \Omega} (w - \tilde{w}) d\mathcal{H}^{d-1} - \int_{\partial \mathcal{A}} (w - \tilde{w})_h d\mathcal{H}^{d-1} \right|$$

$$\begin{aligned}
&= \left| \int_{\partial\mathcal{A}} (\mathcal{J}_h^{d-1} - 1)(w - \tilde{w})_h d\mathcal{H}^{d-1} \right| \\
(181) \quad &\leq \left| \int_{\partial\mathcal{A}} (\mathcal{J}_h^{d-1} - 1) \left((w - \tilde{w})_h - \int_{\partial\mathcal{A}} (w - \tilde{w})_h d\mathcal{H}^{d-1} \right) d\mathcal{H}^{d-1} \right| \\
&\quad + \frac{1}{\mathcal{H}^{d-1}(\partial\mathcal{A})} \left(\int_{\partial\mathcal{A}} |\mathcal{J}_h^{d-1} - 1| d\mathcal{H}^{d-1} \right) \left| \int_{\partial^* A \cap \Omega} (w - \tilde{w}) d\mathcal{H}^{d-1} - \int_{\partial\mathcal{A}} (w - \tilde{w})_h d\mathcal{H}^{d-1} \right| \\
&\quad + \frac{1}{\mathcal{H}^{d-1}(\partial\mathcal{A})} \left(\int_{\partial\mathcal{A}} |\mathcal{J}_h^{d-1} - 1| d\mathcal{H}^{d-1} \right) \left| \int_{\partial^* A \cap \Omega} (w - \tilde{w}) d\mathcal{H}^{d-1} \right|.
\end{aligned}$$

Furthermore, thanks to the estimates (123)–(124), a Taylor expansion, the assumption (69), and Jensen’s inequality

$$(182) \quad \int_{\partial\mathcal{A}} |\mathcal{J}_h^{d-1} - 1| d\mathcal{H}^{d-1} \leq \tilde{C} \sqrt{\mathcal{H}^{d-1}(\partial\mathcal{A})} \sqrt{E[A, \mu|\mathcal{A}]}.$$

Similarly,

$$(183) \quad \left(\int_{\partial\mathcal{A}} |\mathcal{J}_h^{d-1} - 1|^2 d\mathcal{H}^{d-1} \right)^{\frac{1}{2}} \leq \tilde{C} \sqrt{E[A, \mu|\mathcal{A}]}.$$

Hence, inserting the estimates (182) and (183) back into (181) (the second after an application of Cauchy–Schwarz inequality in the first right hand side term of (181)) and recalling the argument for (180), we may upgrade (181) to

$$\begin{aligned}
&\left| \int_{\partial^* A \cap \Omega} (w - \tilde{w}) d\mathcal{H}^{d-1} - \int_{\partial\mathcal{A}} (w - \tilde{w})_h d\mathcal{H}^{d-1} \right| \\
(184) \quad &\leq \tilde{C} \sqrt{\mathcal{H}^{d-1}(\partial\mathcal{A})} C_{PS}(\mathcal{A}) E_{rel}^{\frac{1}{2}}[A, \mu|\mathcal{A}] \left(\int_{\Omega} |\nabla(w - \tilde{w})|^2 dx \right)^{\frac{1}{2}} \\
&\quad + \tilde{C} \frac{\sqrt{E[A, \mu|\mathcal{A}]}}{\sqrt{\mathcal{H}^{d-1}(\partial\mathcal{A})}} \left| \int_{\partial^* A \cap \Omega} (w - \tilde{w}) d\mathcal{H}^{d-1} - \int_{\partial\mathcal{A}} (w - \tilde{w})_h d\mathcal{H}^{d-1} \right| \\
&\quad + \tilde{C} \frac{\sqrt{E[A, \mu|\mathcal{A}]}}{\sqrt{\mathcal{H}^{d-1}(\partial\mathcal{A})}} \left| \int_{\partial^* A \cap \Omega} (w - \tilde{w}) d\mathcal{H}^{d-1} \right|.
\end{aligned}$$

Overall, it follows now from (184), (125), and assumption (71) that for $M \gg 1$

$$\begin{aligned}
(185) \quad II'' &\leq \tilde{C} C_{PS}(\mathcal{A}) E_{rel}[A, \mu|\mathcal{A}] \left(\int_{\Omega} |\nabla(w - \tilde{w})|^2 dx \right)^{\frac{1}{2}} \\
&\quad + \frac{1}{4} \left| \int_{\partial^* A \cap \Omega} (w - \tilde{w}) d\mathcal{H}^{d-1} \right|.
\end{aligned}$$

It follows also immediately from (125), and assumption (71) that for $M \gg 1$

$$(186) \quad III'' \leq \frac{1}{4} \left| \int_{\partial^* A \cap \Omega} (w - \tilde{w}) d\mathcal{H}^{d-1} \right|.$$

Hence, plugging the estimates (180), (185) and (186) back into (179), we obtain as an upgrade of (179)

$$(187) \quad |II| \leq \tilde{C} \sqrt{\mathcal{H}^{d-1}(\partial\mathcal{A})} C_{PS}(\mathcal{A}) E_{rel}^{\frac{1}{2}}[A, \mu|\mathcal{A}] \left(\int_{\Omega} |\nabla(w - \tilde{w})|^2 dx \right)^{\frac{1}{2}}$$

$$\begin{aligned}
& + \tilde{C} C_{PS}(\mathcal{A}) E_{rel}[A, \mu|_{\mathcal{A}}] \left(\int_{\Omega} |\nabla(w-\tilde{w})|^2 dx \right)^{\frac{1}{2}} \\
& + \frac{1}{2} \left| \int_{\partial^* A \cap \Omega} (w-\tilde{w}) d\mathcal{H}^{d-1} \right|.
\end{aligned}$$

In view of (178) and assumption (71), it therefore remains to choose $M \gg 1$ to conclude with (132) in terms of the average $\int_{\partial^* A \cap \Omega} w - \tilde{w} d\mathcal{H}^{d-1}$.

Step 2: Proof of (132), Part II. Decomposing

$$\begin{aligned}
\int_{\partial \mathcal{A}} (w - \tilde{w})_h d\mathcal{H}^{d-1} &= \int_{\partial^* A \cap \Omega} (w - \tilde{w}) d\mathcal{H}^{d-1} \\
&+ \int_{\partial \mathcal{A}} \left((w - \tilde{w}) - \int_{\partial \mathcal{A}} (w - \tilde{w}) d\mathcal{H}^{d-1} \right) (1 - \mathcal{J}_h^{d-1}) d\mathcal{H}^{d-1} \\
&+ \left(\int_{\partial \mathcal{A}} (w - \tilde{w}) d\mathcal{H}^{d-1} \right) \int_{\partial \mathcal{A}} (1 - \mathcal{J}_h^{d-1}) d\mathcal{H}^{d-1},
\end{aligned}$$

we may infer the asserted estimate (132) for $\int_{\partial \mathcal{A}} (w - \tilde{w})_h d\mathcal{H}^{d-1}$ from the corresponding estimate for $\int_{\partial^* A \cap \Omega} (w - \tilde{w}) d\mathcal{H}^{d-1}$ from Step 1 and the arguments used to derive (187).

Step 3: Proof of (134). First, we simply recognize that by means of (50)–(52)

$$\int_{\partial^* A \cap \Omega} \mathbf{n}_{\partial^* A} \cdot \llbracket \nabla \tilde{w} \rrbracket d\mathcal{H}^{d-1} = 0.$$

Second, by assumption (26), the area formula (112) (abbreviating in the following the associated coarea factor by \mathcal{C}_h^{d-1}), and the splitting $\mathbf{n}_{\partial^* \mathcal{A}} = (\mathbf{n}_{\partial^* \mathcal{A}} \cdot \bar{\mathbf{n}}_{\partial \mathcal{A}}) \bar{\mathbf{n}}_{\partial \mathcal{A}} + (\text{Id} - \bar{\mathbf{n}}_{\partial \mathcal{A}} \otimes \bar{\mathbf{n}}_{\partial \mathcal{A}}) \mathbf{n}_{\partial^* \mathcal{A}}$, we obtain

$$\begin{aligned}
\int_{\partial^* A \cap \Omega} \mathbf{n}_{\partial^* A} \cdot B d\mathcal{H}^{d-1} &= \int_{\partial \mathcal{A}} \bar{\mathbf{n}}_{\partial \mathcal{A}} \cdot \llbracket \nabla \tilde{u} \rrbracket \mathcal{C}_h^{d-1} d\mathcal{H}^{d-1} \\
&= \int_{\partial \mathcal{A}} \bar{\mathbf{n}}_{\partial \mathcal{A}} \cdot \llbracket \nabla \tilde{u} \rrbracket (\mathcal{C}_h^{d-1} - 1) d\mathcal{H}^{d-1}.
\end{aligned}$$

Hence,

$$\left| \int_{\partial^* A \cap \Omega} \mathbf{n}_{\partial^* A} \cdot B d\mathcal{H}^{d-1} \right| \leq \tilde{C} \|\mathbf{n}_{\partial \mathcal{A}} \cdot \llbracket \nabla \tilde{u} \rrbracket\|_{L^\infty(\partial \mathcal{A})} \sqrt{\mathcal{H}^{d-1}(\partial \mathcal{A})} \sqrt{E_{vol}[A|_{\mathcal{A}}]},$$

so that the claim (134) follows from the previous three displays. \square

Proof of Lemma 16. By a change of variables Ψ^h , it follows from (50)–(52) and (111)

$$(188) \quad -\nabla \cdot (a^h \nabla \tilde{w}_h) = 0 \quad \text{in } \Omega \setminus \partial \mathcal{A},$$

$$(189) \quad \text{tr}_{\partial \mathcal{A}} \tilde{w}_h = -(\nabla \cdot \xi) \circ (\Psi^h)^{-1} \quad \text{on } \partial \mathcal{A} \cap \Omega,$$

$$(190) \quad (\mathbf{n}_{\partial \Omega} \cdot \nabla) \tilde{w}_h = 0 \quad \text{on } \partial \Omega.$$

Hence, the regularity estimate (135) is just a simple consequence of standard Schauder theory applied to the two regular open sets $\Omega \setminus \overline{\mathcal{A}}$ and \mathcal{A} (see, e.g., [10, Theorem 5.21]). The corresponding estimates depend on the respective domain, the ellipticity constant of a^h , the $C^{0,1/2}$ Hölder norm of a^h on the closure of the respective domain, and finally the $C^{1,1/2}$ Hölder norm of the Dirichlet data $-(\nabla \cdot \xi) \circ (\Psi^h)^{-1}$. We therefore only need to ensure that this data is bounded by a constant of required form $C = C(\mathcal{A}, \Lambda)$. However, recalling

the formulas (116), (117) and $a^h = \frac{1}{|\det \Psi^h|} (\nabla \Psi^h)^\top \nabla \Psi^h$, this claim follows since $\|h\|_{C^{1,1/2}(\partial \mathcal{A})} \leq C(\mathcal{A}, \Lambda)$ due to (44). \square

Proof of Lemma 17. The estimate (140) follows from the chain rule and assumption (69). The estimate (139) in turn follows from the decomposition $(w - \tilde{w})_h = v_h^{(1)} + v_h^{(2)}$ and the energy estimate for $v_h^{(2)}$ satisfying the PDE

$$(191) \quad \Delta v_h^{(2)} = \nabla \cdot ((\text{Id} - a^h) \nabla (w - \tilde{w})_h) \quad \text{in } \Omega \setminus \partial \mathcal{A},$$

with boundary conditions

$$(192) \quad \text{tr}_{\partial \mathcal{A}} v_h^{(2)} = 0 \quad \text{on } \partial \mathcal{A},$$

$$(193) \quad (\mathbf{n}_{\partial \Omega} \cdot \nabla) v_h^{(2)} = 0 \quad \text{on } \partial \Omega.$$

For a proof of (141), we first note that by definition of a^h it holds $\text{supp}(\text{Id} - a^h) \subset B_{\ell/2}(\partial \mathcal{A})$. Hence, by means of the coarea formula (112), we deduce (141) from Cauchy–Schwarz inequality, (135), $|\text{Id} - a^h| \leq \tilde{C}(|\nabla_{\partial \mathcal{A}} h| + \max_{i=1, \dots, d-1} |\kappa_{\partial \mathcal{A}}^i h|)$ in $B_{\ell/2}(\partial \mathcal{A})$, and the estimates (123)–(124). \square

Proof of Lemma 18. We argue by contradiction that there exists $C = C(\mathcal{A}, \Omega)$ such that for all $f \in H^2(\partial \mathcal{A})$ with $\int_{\partial \mathcal{A}} f d\mathcal{H}^{d-1} = 0$ it holds

$$(194) \quad \|f\|_{H_{MS}^{1/2}(\partial \mathcal{A})} \leq C \|f\|_{[H^1(\partial \mathcal{A}), H^2(\partial \mathcal{A})]_{\frac{1}{2}}},$$

where $[H^1, H^2(\partial \mathcal{A})]_{\frac{1}{2}} = H^{\frac{3}{2}}(\partial \mathcal{A})$ denotes the (complex) interpolation space with parameter $\theta = \frac{1}{2}$ between $H^1(\partial \mathcal{A})$ and $H^2(\partial \mathcal{A})$. Of course, (194) implies the claim of Lemma 18.

For each $k \in \mathbb{N}$, assume that we may find a map $f_k \in H^2(\partial \mathcal{A})$ satisfying $\int_{\partial \mathcal{A}} f_k d\mathcal{H}^{d-1} = 0$ such that

$$(195) \quad \|\nabla u_k\|_{L^2(\Omega)} = \|f_k\|_{H_{MS}^{1/2}(\partial \mathcal{A})} = 1 > k \|f_k\|_{[H^1(\partial \mathcal{A}), H^2(\partial \mathcal{A})]_{\frac{1}{2}}},$$

where $u_k \in H^1(\Omega)$ denotes the associated chemical potential, i.e.,

$$\begin{aligned} \Delta u_k &= 0 && \text{in } \Omega \setminus \partial \mathcal{A}, \\ u_k &= f_k && \text{on } \partial \mathcal{A}, \\ (\mathbf{n}_{\partial \Omega} \cdot \nabla) u_k &= 0 && \text{on } \partial \Omega. \end{aligned}$$

We deduce from the elliptic regularity estimate

$$\|u_k\|_{H^2(\Omega \setminus \partial \mathcal{A})} \lesssim_{\mathcal{A}, \Omega} \|u_k\|_{H^1(\Omega)} + \|f_k\|_{[H^1(\partial \mathcal{A}), H^2(\partial \mathcal{A})]_{\frac{1}{2}}}$$

and Poincaré inequality (recall that $\int_{\partial \mathcal{A}} u_k d\mathcal{H}^{d-1} = 0$ by construction) that

$$\sup_{k \in \mathbb{N}} \|u_k\|_{H^2(\Omega \setminus \partial \mathcal{A})} \lesssim_{\mathcal{A}, \Omega} 1.$$

Hence, modulo taking a subsequence, $(u_k)_k$ weakly converges in $H^2(\Omega \setminus \partial \mathcal{A})$ to some $u \in H^2(\Omega \setminus \partial \mathcal{A}) \cap H^1(\Omega)$ such that $u_k \rightarrow u$ strongly in $H^1(\Omega)$. However, by (195), $f_k \rightarrow 0$ strongly in $[H^1(\partial \mathcal{A}), H^2(\partial \mathcal{A})]_{\frac{1}{2}}$, so that u satisfies

$$\begin{aligned} \Delta u &= 0 && \text{in } \Omega \setminus \partial \mathcal{A}, \\ u &= 0 && \text{on } \partial \mathcal{A}, \\ (\mathbf{n}_{\partial \Omega} \cdot \nabla) u &= 0 && \text{on } \partial \Omega. \end{aligned}$$

Hence, $u \equiv 0$ in contradiction to $\lim_{k \rightarrow \infty} \|\nabla u_k\|_{L^2(\Omega)} = 1 = \|\nabla u\|_{L^2(\Omega)}$. \square

Proof of Proposition 8, Part II: Estimates (73) and (74). The argument is naturally a very close variant of the argument in favor of (72). We leave details to the interested reader. \square

6. PROOF OF PROPOSITION 5: REDUCTION TO PERTURBATIVE GRAPH SETTING

6.1. Strategy for the proof of Proposition 5. The proof of Proposition 5 is divided into several steps which are collected and explained here. The corresponding proofs are presented in the next subsection.

First, let us fix the setting for the whole section. Fix $T' \in (0, T_*)$, $C \in (1, \infty)$, and let $\Lambda = \Lambda(\mathcal{A}, A(0), T') \in (0, \infty)$ the constant from Lemma 4. We aim to find $M = M(\Omega, A(0), \mathcal{A}, C, \Lambda, T') \in (1, \infty)$ such that for a.e. $t \in \mathcal{T}_{\text{good}}(\Lambda, M) \cap (0, T')$, i.e.,

$$(196) \quad \int_{\Omega} \frac{1}{2} |\nabla w(\cdot, t)|^2 dx \leq \Lambda,$$

$$(197) \quad E[A, \mu | \mathcal{A}](t) \leq \frac{1}{M} \ell(t)^{d-1},$$

the conclusions of Proposition 5 hold true, see (44)–(47). For what follows, we restrict ourselves to the subset of full measure in $(0, T')$ such that all the a.e. properties of Definition 1 (in particular, the ones of [15, Definition 3]) are satisfied, and fix $t \in (0, T')$ being an element of this subset. It will be convenient to trivially extend the oriented varifold μ_t to an element of $\mathbf{M}(\mathbb{R}^d \times \mathbb{S}^{d-1})$ as well as the vector field $\xi(\cdot, t)$ to a map $\mathbb{R}^d \rightarrow \mathbb{R}^d$ by defining

$$(198) \quad |\mu_t|_{\mathbb{S}^{d-1}} := 0 \text{ and } \xi(\cdot, t) := 0 \quad \text{in } \mathbb{R}^d \setminus \bar{\Omega}.$$

For readability, we suppress from now on the dependence of all quantities on t in the notation.

Heuristically, the idea for our proof of Proposition 5 is based on the competition between two effects:

- On one side, the assumptions (196)–(197) will allow us to prove that around every point $x_0 \in \text{supp } \mu$, the rectifiable set $\text{supp } \mu$ can be represented as a graph within a ball $B_\rho(x_0)$ (parametrized over a suitable affine subspace), where crucially the scale ρ can be chosen independently of a given point $x_0 \in \text{supp } \mu$ (more precisely, $\rho \ll \ell$ uniformly on $[0, T']$). Naturally, the important building block here is Allard's regularity theory, and the assumptions (196)–(197) (in particular, the tuning of the constant M) are used to show that one is in the corresponding setting.
- On the other side, based on the coercivity of the overall error $E[A, \mu | \mathcal{A}]$, any unwanted feature of $\text{supp } \mu$ contradicting the assertions of Proposition 5 (e.g., non-graphical components of $\partial^* A$ as screened from $\partial \mathcal{A}$ in normal direction, or in our case also intersections of $\partial^* A$ with $\partial \Omega$) can be shown to be of arbitrarily small mass, contradicting a lower bound on the mass which one may derive from the local graph property at scale ρ .

For the rigorous argument, we ensure in a first step that our assumption (196) implies a curvature bound.

Lemma 19 (Curvature estimate up to the boundary). *There exists $H: \mathbb{R}^d \rightarrow \mathbb{R}^d \in L^1_{\text{loc}}(\mathbb{R}^d; d|\mu|_{\mathbb{S}^{d-1}})$, $\text{supp } H \subset \bar{\Omega}$, being the generalized mean curvature vector of μ*

with respect to tangential variations, i.e., it holds

$$(199) \quad \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} (\text{Id} - \mathbf{p} \otimes \mathbf{p}) : \nabla B \, d\mu(\cdot, \mathbf{p}) = - \int_{\mathbb{R}^d} \mathbf{H} \cdot B \, d|\mu|_{\mathbb{S}^{d-1}}$$

for all $B \in C^1(\bar{\Omega}; \mathbb{R}^d)$ with $n_{\partial\Omega} \cdot B \equiv 0$ along $\partial\Omega$. Furthermore, for each

$$p \in \begin{cases} [1, \infty) & \text{if } d = 2, \\ [1, 4] & \text{if } d = 3, \end{cases}$$

and $\Lambda \in (0, \infty)$, there exists

$$(200) \quad C_\Lambda := C(\Omega, p, A(0), T, \Lambda) \in (0, \infty)$$

such that the assumption (196) implies

$$(201) \quad \|\mathbf{H}\|_{L^p(\mathbb{R}^d; |\mu|_{\mathbb{S}^{d-1}})} \leq \|w\|_{L^p(\bar{\Omega}; |\mu|_{\mathbb{S}^{d-1}})} \leq C_\Lambda.$$

Based on this, our argument then relies on Allard's regularity theory for integer rectifiable varifolds with free boundary (cf. [29] and [11]). This theory is written down in the language of general varifolds and not oriented ones. However, for the oriented varifold $\mu \in \mathbf{M}(\mathbb{R}^d \times \mathbb{S}^{d-1})$ one may always canonically associate a general varifold $\hat{\mu} \in \mathbf{M}(\mathbb{R}^d \times \mathbf{G}(d, d-1))$, where $\mathbf{G}(d, d-1)$ denotes the space of $(d-1)$ -dimensional linear subspaces of \mathbb{R}^d , by means of

$$(202) \quad \int_{\mathbb{R}^d \times \mathbf{G}(d, d-1)} \varphi(x, \hat{T}) \, d\hat{\mu}(x, \hat{T}) := \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \varphi(x, \hat{T}_{\mathbf{p}}) \, d\mu(x, \mathbf{p}),$$

$$\varphi \in C_{cpt}(\mathbb{R}^d \times \mathbf{G}(d, d-1)),$$

where for $\mathbf{p} \in \mathbb{S}^{d-1}$ we denote by $\hat{T}_{\mathbf{p}} \in \mathbf{G}(d, d-1)$ the 1-dimensional linear subspace of \mathbb{R}^d having \mathbf{p} as its unit normal vector. Note that the mass measure $|\hat{\mu}|_{\mathbf{G}(d, d-1)} \in \mathbf{M}(\mathbb{R}^d)$ of $\hat{\mu}$ is simply $|\mu|_{\mathbb{S}^{d-1}}$ and that the map \mathbf{H} from the previous lemma is the generalized mean curvature vector of $\hat{\mu}$ in the usual sense (with respect to tangential variations). Hence, for what follows we will be cavalier about the distinction between μ and $\hat{\mu}$.

Theorem 20 (Allard's regularity theory—interior and boundary case). *Fix data $\rho \in (0, \ell)$, $x_0 \in \text{supp } \mu$, and $\hat{T} \in \mathbf{G}(d, d-1)$.*

There exist constants $\varepsilon_{reg}, \gamma_{reg} \in (0, 1)$ and $C_{reg} \in (1, \infty)$ such that:

i) If $x_0 \in \text{supp } \mu \cap \Omega$ such that (199) holds for all $B \in C_{cpt}^1(B_{2\rho}(x_0); \mathbb{R}^d)$ and

$$(203) \quad \begin{cases} \frac{|\mu|_{\mathbb{S}^{d-1}}(B_\rho(x_0))}{\omega_{d-1}\rho^{d-1}} \leq 1 + \varepsilon_{reg}, \\ \left\{ E_*^\circ[x_0, \rho, \hat{T}] := \max \left\{ E_{\text{tilt}}[x_0, \rho, \hat{T}], \frac{\rho^{2(1-\frac{d-1}{4})}}{\varepsilon_{reg}} \left(\int_{B_\rho(x_0)} |\mathbf{H}|^4 \, d|\mu|_{\mathbb{S}^{d-1}} \right)^{\frac{1}{2}} \right\} \leq \varepsilon_{reg}, \end{cases}$$

where $E_{\text{tilt}}[x_0, \rho, \hat{T}]$ is the usual tilt excess relative to the integer rectifiable varifold $\hat{\mu}$, then there exists a $C^{1, 1-\frac{d-1}{4}}$ function

$$(204) \quad u: (x_0 + \hat{T}) \cap B_{\gamma_{reg}\rho}(x_0) \rightarrow \mathbb{R}$$

with the following properties: $u(x_0) = 0$,

$$(205) \quad \begin{aligned} & \text{supp } \mu \cap B_{\gamma_{reg}\rho}(x_0) \\ &= \{y + u(y)n_{\hat{T}}(y) : y \in (x_0 + \hat{T}) \cap B_{\gamma_{reg}\rho}(x_0)\} \cap B_{\gamma_{reg}\rho}(x_0), \end{aligned}$$

where $\mathbf{n}_{\widehat{T}}$ is a normal vector of the affine subspace $x_0 + \widehat{T}$, and

$$(206) \quad \begin{aligned} & \left\| (\rho^{-1}u, \nabla_{x_0 + \widehat{T}}u) \right\|_{L^\infty((x_0 + \widehat{T}) \cap B_{\gamma_{reg}\rho}(x_0))} \\ & + \rho^{1 - \frac{d-1}{4}} [\nabla_{x_0 + \widehat{T}}u]_{C^{0,1-\frac{d-1}{4}}((x_0 + \widehat{T}) \cap B_{\gamma_{reg}\rho}(x_0))} \\ & \leq C_{reg} \left\{ E_{\tilde{t}ilt}^{\frac{1}{2}}[x_0, \rho, \widehat{T}] + \rho^{1 - \frac{d-1}{4}} \left(\int_{B_\rho(x_0)} |\mathbf{H}|^4 d|\mu|_{\mathbb{S}^{d-1}} \right)^{\frac{1}{4}} \right\}. \end{aligned}$$

ii) If $x_0 \in \text{supp } \mu \cap \partial\Omega$ such that

$$(207) \quad \begin{cases} \frac{|\mu|_{\mathbb{S}^{d-1}}(B_\rho(x_0)) + |\mu|_{\mathbb{S}^{d-1}}(\tilde{B}_\rho(x_0))}{\omega_{d-1}\rho^{d-1}} \leq 1 + \varepsilon_{reg}, \\ E_*[x_0, \rho, \text{Tan}_{x_0}^\perp \partial\Omega] := \max \left\{ E_*^\circ[x_0, \rho, \text{Tan}_{x_0}^\perp \partial\Omega], \frac{\rho}{\ell} \right\} \leq \varepsilon_{reg}, \end{cases}$$

where $\tilde{B}_\rho(x_0) := \{x \in \mathbb{R}^d : |\tilde{x} - x_0| < \rho \text{ where } \tilde{x} := x - 2s_{\partial\Omega}(x)\mathbf{n}_{\partial\Omega}(P_{\partial\Omega}(x))\}$, then there exists a $C^{1, \frac{1}{2}}$ function

$$(208) \quad u : (x_0 + \widehat{T}) \cap B_{\gamma_{reg}\rho}(x_0) \rightarrow \mathbb{R}$$

with the following properties: $u(x_0) = 0$,

$$(209) \quad \begin{aligned} & \text{supp } \mu \cap B_{\gamma_{reg}\rho}(x_0) \\ & = \{y + u(y)\mathbf{n}_{\widehat{T}}(y) : y \in (x_0 + \widehat{T}) \cap B_{\gamma_{reg}\rho}(x_0)\} \cap B_{\gamma_{reg}\rho}(x_0) \cap \bar{\Omega}, \end{aligned}$$

and

$$(210) \quad \mathbf{n}_{\partial\Omega}(x_0) \in \text{Tan}_{x_0}(\text{supp } \mu)$$

as well as

$$(211) \quad \begin{aligned} & \left\| (\rho^{-1}u, \nabla_{x_0 + \widehat{T}}u) \right\|_{L^\infty((x_0 + \widehat{T}) \cap B_{\gamma_{reg}\rho}(x_0) \cap \bar{\Omega})} \\ & \leq C_{reg} \varepsilon_{reg}^{\frac{1}{2(d-1)+3}}, \\ & \rho^{\frac{1}{2}} [\nabla_{x_0 + \widehat{T}}u]_{C^{0, \frac{1}{2}}((x_0 + \widehat{T}) \cap B_{\gamma_{reg}\rho}(x_0) \cap \bar{\Omega})} \\ & \leq C_{reg} \left\{ E_{\tilde{t}ilt}^{\frac{1}{2}}[x_0, \rho, \widehat{T}] + \rho^{1 - \frac{d-1}{4}} \left(\int_{B_\rho(x_0)} |\mathbf{H}|^4 d|\mu|_{\mathbb{S}^{d-1}} \right)^{\frac{1}{4}} + \left(\frac{\rho}{\ell} \right)^{\frac{1}{2}} \right\}. \end{aligned}$$

In a next step, we aim to define a scale $\rho_{reg} \in (0, \ell)$, uniformly on $[0, T']$, such that one may afterward choose $M \gg 1$, again uniformly on $[0, T']$, so that the assumptions (196)–(197) imply, among other things, that for all $x_0 \in \text{supp } |\mu|_{\mathbb{S}^{d-1}}$, the rectifiable set $\text{supp } |\mu|_{\mathbb{S}^{d-1}}$ can be locally represented around x_0 as a graph on scale $\gamma_{reg}\rho_{reg}$ (with respect to a suitably chosen subspace) in the precise sense of Theorem 20.

By locally uniform regularity of \mathcal{A} , one may choose

$$(212) \quad \varepsilon = \varepsilon(\mathcal{A}) \in \left(0, \min \left\{ \varepsilon_{reg}, \frac{1}{8} \frac{1}{C_{reg}} \frac{1}{16C} \right\} \right) \quad \text{and} \quad \tilde{C} = \tilde{C}(\mathcal{A}) \in (1, \infty)$$

uniformly on $[0, T']$ such that for all

$$(213) \quad \rho \in (0, \rho_{reg}], \quad \text{where } \rho_{reg} := \tilde{C}^{-1}\tilde{\rho} \text{ and } \tilde{\rho} := \frac{\varepsilon^2}{C_\Lambda^2} \ell,$$

at least the following properties are satisfied ($P_{\text{Tan}_y \partial \mathcal{A}}$ denotes the orthogonal projection onto the tangent space $\text{Tan}_y \partial \mathcal{A}$ at $y \in \partial \mathcal{A}$):

- it holds

$$(214) \quad x_0 \in \{|\xi| \leq 1/2\} \implies B_\rho(x_0) \subset \{|\xi| \leq 3/4\},$$

- for all $x_0 \in \{|\xi| > \frac{1}{2}\}$ and all $x \in B_\rho(x_0)$ it holds

$$(215) \quad |\xi(x) - \mathbf{n}_{\partial\mathcal{A}}(P_{\partial\mathcal{A}}(x_0))| \leq \frac{1}{1 + \varepsilon/2} \frac{\varepsilon^2}{4},$$

- for all $x_0 \in \{|\xi| > \frac{1}{2}\}$ and all $y \in B_{\tilde{C}^{-1}\rho}(y_0) \cap \partial\mathcal{A}$, where $y_0 := P_{\partial\mathcal{A}}(x_0)$, it holds

$$(216) \quad \begin{aligned} &P_{x_0 + \text{Tan}_{y_0} \partial\mathcal{A}}(B_\rho(x_0) \cap \{y + s\mathbf{n}_{\partial\mathcal{A}}(y) : s \in (-\ell, \ell)\}) \\ &\subset (x_0 + \text{Tan}_{y_0} \partial\mathcal{A}) \cap B_{(\cos \alpha_{reg})\rho}(x_0) \end{aligned}$$

and

$$(217) \quad \text{the opening angle between } \mathbf{n}_{\partial\mathcal{A}}(y) \text{ and } \mathbf{n}_{\partial\mathcal{A}}(y_0) \text{ is at most } \frac{1}{2} \left(\frac{\pi}{2} - \alpha_{reg} \right),$$

where $\alpha_{reg} \in (0, \frac{\pi}{2})$ is defined by $3C_{reg}\varepsilon =: \tan \alpha_{reg}$,

- for all $y_0, y \in \partial\mathcal{A}$ it holds

$$(218) \quad |P_{\text{Tan}_{y_0} \partial\mathcal{A}}(y - y_0)| \leq \rho \implies |(y - y_0) \cdot \mathbf{n}_{\partial\mathcal{A}}(y_0)| \leq \frac{1}{4} \left(\frac{\ell}{16C} \right).$$

Condition (214) will be applied to show that $\text{supp } \mu$ cannot be located too far away from $\partial\mathcal{A}$. Condition (215) will be of importance whenever one has to switch between the usual geometric measure theory tilt excess E_{tilt} and our relative entropy E_{rel} , as E_{tilt} is working locally with a fixed tangent space whereas E_{rel} encodes tilt excess at each point based on the varying vector field ξ . We will rely on conditions (215) and (216) for the proof of the graph property (45); more precisely, if $x_0 \in \text{supp } \mu$ is a point sufficiently close to $\partial\mathcal{A}$ where $\text{supp } \mu$ admits the graph representation (205)–(206) with respect to $\hat{T} = \text{Tan}_{P_{\partial\mathcal{A}}(x_0)} \partial\mathcal{A}$ locally on scale $\rho = \gamma_{reg}\rho_{reg}$, then (215) and (216) ensure that $\text{supp } \mu$ can also be written as a graph in the sense of (45) locally on the smaller scale $\tilde{C}^{-1}\gamma_{reg}\rho_{reg}$. Finally, condition (218) is just a local height estimate for the interface $\partial\mathcal{A}$ if one locally screens it from $y_0 + \text{Tan}_{y_0}(\partial\mathcal{A})$, and it will be needed in our proof for the estimate (47).

With these choices in place, we first show that ρ_{reg} indeed satisfies the aforementioned goal of representing $\text{supp } |\mu|_{\mathbb{S}^{d-1}}$ locally on scale $\gamma_{reg}\rho_{reg}$ around any of its points as a graph.

Lemma 21 (Applicability of Allard regularity theory). *There exists $M_0 \gg_{\mathcal{A}, C_\Lambda} 1$, uniformly on $[0, T']$, such that for all $M \geq M_0$ the assumptions (196)–(197) imply that the hypotheses (203) hold true at scale $\rho = \rho_{reg}$ for all interior points $x_0 \in \text{supp } \mu \cap \Omega$ and*

$$(219) \quad \begin{cases} \text{any } \hat{T} \in \mathbf{G}(d, d-1) & \text{if } x_0 \in \{|\xi| \leq 1/2\}, \\ \hat{T} = \text{Tan}_{P_{\partial\mathcal{A}}(x_0)} \partial\mathcal{A} & \text{if } x_0 \in \{|\xi| > 1/2\}, \end{cases}$$

as well as that the hypotheses (207) hold true at scale $\rho = \rho_{reg}$ for all boundary points $x_0 \in \text{supp } \mu \cap \partial\Omega$ (and $\hat{T} \in \mathbf{G}(d, d-1)$ fixed by $\text{Tan}_{x_0}^\perp \partial\Omega$). Furthermore, in both cases

$$(220) \quad E_{tilt}^{\frac{1}{2}}[x_0, \rho_{reg}, \hat{T}] \leq \varepsilon,$$

$$(221) \quad \rho_{reg}^{1-\frac{d-1}{4}} \left(\int_{B_{\rho_{reg}}(x_0)} |\mathbf{H}|^4 d|\mu|_{\mathbb{S}^{d-1}} \right)^{\frac{1}{4}} \leq \varepsilon$$

for all $x_0 \in \text{supp } \mu$ with associated data $(\rho_{reg}, \widehat{T})$ as above.

By means of the previous result, we then ensure that the mass measure of the varifold μ reduces to the mass measure of the reduced boundary of the finite perimeter set A and that the latter is supported sufficiently close to the interface $\partial\mathcal{A}$ of the strong solution (in particular, of positive distance to the physical boundary $\partial\Omega$).

Lemma 22 (Reduction to mass measure with no boundary intersection). *One may choose $M_0 \gg_{\mathcal{A}, C_\Lambda} 1$ uniformly on $[0, T']$ such that for all $M \geq M_0$ the assumptions (196)–(197) imply*

$$(222) \quad \partial^* A \subset \{|\xi| > 1/2\} \subset B_\ell(\partial\mathcal{A}),$$

$$(223) \quad |\mu|_{\mathbb{S}^{d-1}} = \mathcal{H}^{d-1} \llcorner \partial^* A$$

and

$$(224) \quad \begin{aligned} \int_{\partial^* A} (\text{Id} - \mathbf{n}_{\partial^* A} \otimes \mathbf{n}_{\partial^* A}) : \nabla B d\mathcal{H}^{d-1} &= \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} (\text{Id} - s \otimes s) : \nabla B d\mu \\ &= - \int_{\mathbb{R}^d} \mathbf{H} \cdot B d|\mu|_{\mathbb{S}^{d-1}} \end{aligned}$$

for all $B \in C_{cpt}^1(\mathbb{R}^d; \mathbb{R}^d)$.

In the next step, we leverage on the previous results to show that the interface of the weak solution $\partial^* A$ is in fact “sufficiently rich” in order to allow for a graph representation relative to the interface of the strong solution $\partial\mathcal{A}$. Note that this cannot follow just from requiring smallness of the relative entropy (17)—which, by careful inspection of the proofs, is in fact up to now the only required ingredient concerning the smallness of the overall error—but also requires smallness of the volume error (18): indeed, the relative entropy provides no error control in the regime of vanishing interfacial mass $|\mu|_{\mathbb{S}^{d-1}}(\mathbb{R}^d) \downarrow 0$.

Lemma 23 (Graph representation). *One may choose $M_0 \gg_{\mathcal{A}, C_\Lambda} 1$ uniformly on $[0, T']$ such that for all $M \geq M_0$ the assumptions (196)–(197) imply that $\partial^* A$ can be represented as a graph over $\partial\mathcal{A}$ in the sense of (45) with regularity (44).*

We finally conclude by showing that the already established graph representation satisfies all the required properties.

Lemma 24 (Height function estimates). *One may choose $M_0 \gg_{\mathcal{A}, C_\Lambda} 1$ uniformly on $[0, T']$ such that for all $M \geq M_0$ the assumptions (196)–(197) imply that the height function $h: \partial\mathcal{A} \rightarrow [-\ell, \ell]$ from Lemma 23 representing $\partial^* A$ as a graph over $\partial\mathcal{A}$ satisfies the estimates (47).*

6.2. Proofs. We continue with the proofs of the several intermediate results from the previous subsection, eventually culminating into a proof of Proposition 5, which itself is the last missing ingredient for the proof of our main result, Theorem 1.

Proof of Lemma 19. Denote by $H_{|\mu|_{\mathbb{S}^{d-1}} \llcorner \Omega}: \text{supp}(|\mu|_{\mathbb{S}^{d-1}} \llcorner \Omega) \rightarrow \mathbb{R}^d$ the map from [15, Definition 3, item iii)] and define

$$(225) \quad \mathbf{H} := \begin{cases} H_{|\mu|_{\mathbb{S}^{d-1}} \llcorner \Omega} & \text{in } \Omega, \\ 0 & \text{else.} \end{cases}$$

The identity (199) then directly follows from (225) and [15, (17h)], whereas the estimate (201) is a consequence of [15, Proposition 5, (26), Definition 3 item iv)] and (7c). \square

Proof of Theorem 20. We naturally distinguish between two cases.

Case 1: For the case of interior points $x_0 \in \text{supp } \mu \cap \Omega$, let $\bar{\varepsilon}, \bar{\gamma} \in (0, 1)$ and $\bar{c} \in (1, \infty)$ be the constants from [29, 23.1 Theorem]. It follows from [29, 23.2 Remark, item (2)] and our hypotheses that for suitably small $\varepsilon_{reg} \in (0, \bar{\varepsilon})$

$$(226) \quad \frac{|\mu|_{\mathbb{S}^{d-1}}(B_\sigma(x))}{\omega_{d-1}\sigma^{d-1}} \leq \frac{3}{2}$$

for all $\sigma \in (0, \frac{\rho}{2})$ and all $x \in \text{supp } \mu \cap B_{\varepsilon_{reg}\rho}(x_0)$. In particular, the $(d-1)$ -dimensional density of μ , denoted by $\Theta_{d-1}(\mu, \cdot)$, satisfies

$$(227) \quad \Theta_{d-1}(\mu, x) = 1 \quad \text{for all } x \in \text{supp } \mu \cap B_{\varepsilon_{reg}\rho}(x_0).$$

Furthermore, for small enough $\varepsilon_{reg} \in (0, \bar{\varepsilon})$, we deduce from [29, Proof of 23.1, inequality (12)] that

$$(228) \quad E_*^\circ[x_0, (\varepsilon_{reg}\rho)/2, \widehat{T}] \leq \bar{\varepsilon}.$$

Hence, the hypotheses of [29, 23.1 Theorem] are fulfilled for the choices $U = B_{\varepsilon_{reg}\rho}(x_0)$ and $\rho = (\varepsilon_{reg}\rho)/2$, so that the remaining claims follow from the conclusions of [29, 23.1 Theorem] for $\gamma_{reg} := (\varepsilon_{reg}\bar{\gamma})/2$ and $C_{reg} := \bar{c}$.

Case 2: For the case of boundary points $x_0 \in \text{supp } \mu \cap \partial\Omega$, one may argue analogously, using [11, inequality between (40) and (41)], [11, inequality (51)] and [11, 4.9 Theorem] as substitutes for [29, 23.2 Remark, item (2)], [29, Proof of 23.1, inequality (12)] and [29, 23.1 Theorem], respectively. Note that the analogue of (229) is given by

$$(229) \quad \frac{|\mu|_{\mathbb{S}^{d-1}}(B_\sigma(x)) + |\mu|_{\mathbb{S}^{d-1}}(\tilde{B}_\sigma(x))}{\omega_{d-1}\sigma^{d-1}} \leq \frac{3}{2}$$

for all $\sigma \in (0, \frac{\rho}{2})$ and all $x \in \text{supp } \mu \cap B_{\varepsilon_{reg}\rho}(x_0)$. In particular,

$$(230) \quad \begin{aligned} \Theta_{d-1}(\mu, x) &= 1 \quad \text{for all } x \in \text{supp } \mu \cap B_{\varepsilon_{reg}\rho}(x_0) \cap \Omega, \\ \Theta_{d-1}(\mu, x) &\leq \frac{3}{4} \quad \text{for all } x \in \text{supp } \mu \cap B_{\varepsilon_{reg}\rho}(x_0) \cap \partial\Omega; \end{aligned}$$

a fact which we state for future reference (cf. [11, 3.2 Corollary]). Note finally that the second statement of (230) implies $|\mu|_{\mathbb{S}^{d-1}}(B_{\varepsilon_{reg}\rho}(x_0) \cap \partial\Omega) = 0$. \square

Proof of Lemma 21. Using the bound (201) and the definitions (212)–(213), we get

$$(231) \quad \frac{\rho_{reg}^{2(1-\frac{d-1}{4})}}{\varepsilon_{reg}} \left(\int_{B_{\rho_{reg}}(x_0)} |\mathbb{H}|^4 d|\mu|_{\mathbb{S}^{d-1}} \right)^{\frac{1}{2}} \leq \frac{\rho_{reg}}{\varepsilon_{reg}} C_\Lambda^2 \leq \frac{\varepsilon^2}{\varepsilon_{reg}} \leq \varepsilon_{reg}.$$

The penultimate inequality settles (221) whereas the last is precisely the curvature estimate required by (203) and (207). Note also that by definitions (212)–(213) it holds $\rho_{reg}/\ell \leq \varepsilon_{reg}$. It thus remains to derive the asserted estimates for the mass ratios and the tilt excess. To this end, we distinguish between the three natural cases.

Case 1: $x_0 \in \text{supp } \mu \cap \Omega \cap \{|\xi| \leq 1/2\}$. We start estimating by a union bound and recalling (20)–(21)

$$(232) \quad \begin{aligned} |\mu|_{\mathbb{S}^{d-1}}(B_{\rho_{reg}}(x_0)) &\leq |\mu|_{\mathbb{S}^{d-1}}(B_{\rho_{reg}}(x_0) \cap \partial\Omega) \\ &\quad + |\mu|_{\mathbb{S}^{d-1}}(B_{\rho_{reg}}(x_0) \cap \Omega \cap \{\varrho \leq 1/3\}) \\ &\quad + |\mu|_{\mathbb{S}^{d-1}}(B_{\rho_{reg}}(x_0) \cap \Omega \cap \{\varrho = 1\}). \end{aligned}$$

Recalling also (22) and from property (214) that $B_{\rho_{reg}}(x_0) \subset \{|\xi| \leq \frac{3}{4}\}$, we get

$$(233) \quad \begin{aligned} &|\mu|_{\mathbb{S}^{d-1}}(B_{\rho_{reg}}(x_0) \cap \partial\Omega) + |\mu|_{\mathbb{S}^{d-1}}(B_{\rho_{reg}}(x_0) \cap \Omega \cap \{\varrho \leq 1/3\}) \\ &\leq |\mu|_{\mathbb{S}^{d-1}}(\partial\Omega) + \frac{3}{2} \int_{\Omega \cap \{\varrho \leq \frac{1}{3}\}} 1 - \varrho d|\mu|_{\mathbb{S}^{d-1}} \leq \frac{3}{2} E_{rel}[A, \mu|_{\mathcal{A}}] \end{aligned}$$

and

$$(234) \quad \begin{aligned} &|\mu|_{\mathbb{S}^{d-1}}(B_{\rho_{reg}}(x_0) \cap \Omega \cap \{\varrho = 1\}) \\ &\leq \int_{\partial^* A \cap \Omega \cap \{|\xi| \leq 3/4\}} 1 d\mathcal{H}^{d-1} \\ &\leq 4 \int_{\partial^* A \cap \Omega} (1 - \mathbf{n}_{\partial^* A} \cdot \xi) d\mathcal{H}^{d-1} \leq 4E_{rel}[A, \mu|_{\mathcal{A}}]. \end{aligned}$$

Hence, the previous three displays together with assumption (197) imply

$$(235) \quad \frac{|\mu|_{\mathbb{S}^{d-1}}(B_{\rho_{reg}}(x_0))}{\omega_{d-1}\rho_{reg}^{d-1}} \leq \frac{11}{2} \frac{\tilde{C}^{d-1} C_{\Lambda}^{d-1}}{\omega_{d-1}\varepsilon^{2(d-1)}} \frac{1}{M} \stackrel{M \geq M_0 \gg_{\mathcal{A}, C_{\Lambda}} 1}{\leq} \varepsilon^2.$$

Let now $\hat{T} \in \mathbf{G}(d, d-1)$ arbitrary but fixed. We may then analogously ensure by choosing $M \geq M_0 \gg_{\mathcal{A}, C_{\Lambda}} 1$ that

$$(236) \quad E_{tilt}[x_0, \rho_{reg}, \hat{T}] \leq \varepsilon^2$$

because of the simple observation that $E_{tilt}[x_0, \rho_{reg}, \hat{T}] \leq C_{tilt} \frac{|\mu|_{\mathbb{S}^{d-1}}(B_{\rho_{reg}}(x_0))}{\omega_{d-1}\rho_{reg}^{d-1}}$ for some absolute constant $C_{tilt} \in (1, \infty)$.

Case 2: $x_0 \in \text{supp } \mu \cap \partial\Omega$. Due to [17, Lemma 4.2], $\rho_{reg} \leq \ell/8$, (8) and (16), we know that

$$(237) \quad \tilde{B}_{\rho_{reg}}(x_0) \subset B_{5\rho_{reg}}(x_0) \subset B_{\ell}(\partial\Omega) \subset \{|\xi| = 0\}.$$

Hence, arguing essentially analogous to the previous case shows

$$(238) \quad |\mu|_{\mathbb{S}^{d-1}}(B_{\rho_{reg}}(x_0)) + |\mu|_{\mathbb{S}^{d-1}}(\tilde{B}_{\rho_{reg}}(x_0)) \leq 3E_{rel}[A, \mu|_{\mathcal{A}}],$$

so that a suitable choice of $M \geq M_0 \gg_{\mathcal{A}, C_{\Lambda}} 1$ allows to guarantee

$$(239) \quad \frac{|\mu|_{\mathbb{S}^{d-1}}(B_{\rho_{reg}}(x_0)) + |\mu|_{\mathbb{S}^{d-1}}(\tilde{B}_{\rho_{reg}}(x_0))}{\omega_{d-1}\rho_{reg}^{d-1}} \leq \varepsilon^2$$

as well as

$$(240) \quad E_{tilt}[x_0, \rho_{reg}, \text{Tan}_{x_0}^{\perp} \partial\Omega] \leq \varepsilon^2.$$

Case 3: $x_0 \in \text{supp } \mu \cap \Omega \cap \{|\xi| > 1/2\}$. By (232), (233) and (20) it holds

$$(241) \quad |\mu|_{\mathbb{S}^{d-1}}(B_{\rho_{reg}}(x_0)) \leq \frac{3}{2} E_{rel}[A, \mu|_{\mathcal{A}}] + \int_{\partial^* A \cap B_{\rho_{reg}}(x_0)} 1 d\mathcal{H}^{d-1}.$$

Note that indeed $\partial^* A \cap B_{\rho_{reg}}(x_0) = \partial^* A \cap B_{\rho_{reg}}(x_0) \cap \Omega$ since $x_0 \in \Omega \cap \{|\xi| > 1/2\}$ implies $B_{\rho_{reg}}(x_0) \subset B_{\ell}(\partial\mathcal{A}) \subset \Omega$ due to (16), $\rho_{reg} \leq \ell/8$ and (8).

To estimate the second term on the right hand side of (241), first introduce some additional notation. Define $T_{x_0} := P_{\partial\mathcal{A}}(x_0) + \text{Tan}_{P_{\partial\mathcal{A}}(x_0)}\partial\mathcal{A}$ and let P_{x_0} the nearest point projection onto T_{x_0} . For every $x \in \mathbb{R}^d$, we further denote by $A_{P_{x_0}(x)}$ the one-dimensional slice $A \cap \{P_{x_0}(x) + \text{sn}_{\partial\mathcal{A}}(P_{\partial\mathcal{A}}(x_0)) : |s| \leq \ell\}$. Finally, define the sets

$$(242) \quad S_{x_0}^{(1)} := \partial^* A \cap B_{\rho_{reg}}(x_0) \cap \{x \in \mathbb{R}^d : \mathcal{H}^0(\partial^* A_{P_{x_0}(x)}) > 1\},$$

$$(243) \quad S_{x_0}^{(2)} := \left\{ (\partial^* A \cap B_{\rho_{reg}}(x_0)) \setminus S_{x_0}^{(1)} : \mathbf{n}_{\partial^* A}(x) \cdot \xi(x) \geq \frac{1+\varepsilon/4}{1+\varepsilon/2} \right\},$$

$$(244) \quad S_{x_0}^{(3)} := \left\{ (\partial^* A \cap B_{\rho_{reg}}(x_0)) \setminus S_{x_0}^{(1)} : \mathbf{n}_{\partial^* A}(x) \cdot \xi(x) < \frac{1+\varepsilon/4}{1+\varepsilon/2} \right\}.$$

In particular,

$$(245) \quad \partial^* A \cap B_{\rho_{reg}}(x_0) = S_{x_0}^{(1)} \cup S_{x_0}^{(2)} \cup S_{x_0}^{(3)}$$

and, by definition of $S_{x_0}^{(1)}$ and $S_{x_0}^{(2)}$ as well as by the flatness property (215),

$$(246) \quad x \in S_{x_0}^{(2)} \implies \mathcal{H}^0(\partial^* A_{P_{x_0}(x)}) = 1 \text{ and } |\mathbf{n}_{\partial^* A}(x) \cdot \mathbf{n}_{\partial\mathcal{A}}(P_{\partial\mathcal{A}}(x_0))| \geq \frac{1}{1+\varepsilon/2}.$$

Now, we estimate term by term in the decomposition (245). First, a slicing argument ensures

$$(247) \quad \mathcal{H}^{d-1}(S_{x_0}^{(1)}) \leq C_{rel} E_{rel}[A, \mu|_{\mathcal{A}}]$$

for some universal constant $C_{rel} \in (1, \infty)$. Second, one immediately deduces that

$$(248) \quad \mathcal{H}^{d-1}(S_{x_0}^{(3)}) \leq \frac{4(1+\varepsilon/2)}{\varepsilon} \int_{\partial^* A \cap \Omega} (1 - \mathbf{n}_{\partial^* A} \cdot \xi) d\mathcal{H}^{d-1} \leq \frac{4(1+\varepsilon/2)}{\varepsilon} E_{rel}[A, \mu|_{\mathcal{A}}].$$

Third, by coarea formula and (246)

$$(249) \quad \mathcal{H}^{d-1}(S_{x_0}^{(2)}) \leq (1+\varepsilon/2) \mathcal{H}^{d-1}(B_{\rho_{reg}}(P_{\partial\mathcal{A}}(x_0))) = (1+\varepsilon/2) \omega_{d-1} \rho_{reg}^{d-1}.$$

In summary, we may now infer from (241), (245) and (247)–(249) that

$$(250) \quad |\mu|_{\mathbb{S}^{d-1}}(B_{\rho_{reg}}(x_0)) \leq (1+\varepsilon/2) \omega_{d-1} \rho_{reg}^{d-1} + \left(\frac{3}{2} + C_{rel} + \frac{4(1+\varepsilon/2)}{\varepsilon} \right) E_{rel}[A, \mu|_{\mathcal{A}}]$$

for some universal constant $C_{rel} \in (1, \infty)$. Hence, choosing $M \geq M_0 \gg_{\mathcal{A}, C_\Lambda} 1$ in a suitable manner entails by assumption (197)

$$(251) \quad |\mu|_{\mathbb{S}^{d-1}}(B_{\rho_{reg}}(x_0)) \leq (1+\varepsilon) \omega_{d-1} \rho_{reg}^{d-1}.$$

Having established the desired estimate on the mass ratio, we now turn to the asserted bound for the tilt excess and claim that, upon suitably choosing M_0 ,

$$(252) \quad E_{tilt}[x_0, \rho_{reg}, T_{x_0}] \leq \varepsilon^2.$$

The argument in favor of (252) is very close to the one producing (251).

First, employing the decomposition underlying (232), we obtain as the analogue of (241)

$$(253) \quad \begin{aligned} & E_{\text{tilt}}[x_0, \rho_{\text{reg}}, T_{x_0}] \\ & \leq C_{\text{tilt}} \frac{E_{\text{rel}}[A, \mu | \mathcal{A}]}{\rho_{\text{reg}}^{\frac{d-1}{d}}} + \frac{1}{\rho_{\text{reg}}^{\frac{d-1}{d}}} \int_{\partial^* A \cap B_{\rho_{\text{reg}}}(x_0)} (1 - \mathbf{n}_{\partial^* A}(x) \cdot \mathbf{n}_{\partial \mathcal{A}}(P_{\partial \mathcal{A}}(x_0))) d\mathcal{H}^{d-1}(x), \end{aligned}$$

where $C_{\text{tilt}} \in (1, \infty)$ is a universal constant. For an estimate of the second term on the right hand side of the previous display, we again make use of the decomposition (242)–(245). Hence, due to the estimates (247) and (248),

$$(254) \quad \begin{aligned} & \frac{1}{\rho_{\text{reg}}^{\frac{d-1}{d}}} \int_{\partial^* A \cap B_{\rho_{\text{reg}}}(x_0)} (1 - \mathbf{n}_{\partial^* A}(x) \cdot \mathbf{n}_{\partial \mathcal{A}}(P_{\partial \mathcal{A}}(x_0))) d\mathcal{H}^{d-1}(x) \\ & \leq \frac{2}{\rho_{\text{reg}}^{\frac{d-1}{d}}} (\mathcal{H}^{d-1}(S_{x_0}^{(1)}) + \mathcal{H}^{d-1}(S_{x_0}^{(3)})) \\ & \quad + \frac{1}{\rho_{\text{reg}}^{\frac{d-1}{d}}} \int_{S_{x_0}^{(2)}} (1 - \mathbf{n}_{\partial^* A}(x) \cdot \mathbf{n}_{\partial \mathcal{A}}(P_{\partial \mathcal{A}}(x_0))) d\mathcal{H}^{d-1}(x) \\ & \leq 2 \left(C'_{\text{tilt}} + \frac{4(1+\varepsilon/2)}{\varepsilon} \right) \frac{E_{\text{rel}}[A, \mu | \mathcal{A}]}{\rho_{\text{reg}}^{\frac{d-1}{d}}} \\ & \quad + \frac{1}{\rho_{\text{reg}}^{\frac{d-1}{d}}} \int_{S_{x_0}^{(2)}} (1 - \mathbf{n}_{\partial^* A}(x) \cdot \mathbf{n}_{\partial \mathcal{A}}(P_{\partial \mathcal{A}}(x_0))) d\mathcal{H}^{d-1}(x) \end{aligned}$$

for some universal constant $C'_{\text{tilt}} \in (1, \infty)$. Furthermore, adding zero and estimating based on (249) and (215) yields

$$(255) \quad \begin{aligned} & \frac{1}{\rho_{\text{reg}}^{\frac{d-1}{d}}} \int_{S_{x_0}^{(2)}} (1 - \mathbf{n}_{\partial^* A}(x) \cdot \mathbf{n}_{\partial \mathcal{A}}(P_{\partial \mathcal{A}}(x_0))) d\mathcal{H}^{d-1}(x) \\ & \leq \frac{E_{\text{rel}}[A, \mu | \mathcal{A}]}{\rho_{\text{reg}}} + \frac{1}{\rho_{\text{reg}}^{\frac{d-1}{d}}} \int_{S_{x_0}^{(2)}} |\xi(x) - \mathbf{n}_{\partial \mathcal{A}}(P_{\partial \mathcal{A}}(x_0))| d\mathcal{H}^{d-1}(x) \\ & \leq \frac{E_{\text{rel}}[A, \mu | \mathcal{A}]}{\rho_{\text{reg}}^{\frac{d-1}{d}}} + \frac{\varepsilon^2}{4} (1+\varepsilon/2) \omega_{d-1}. \end{aligned}$$

In summary, we obtain from the previous three displays that

$$(256) \quad E_{\text{tilt}}[x_0, \rho_{\text{reg}}, T_{x_0}] \leq 2 \left(C''_{\text{tilt}} + \frac{4(1+\varepsilon/2)}{\varepsilon} \right) \frac{E_{\text{rel}}[A, \mu | \mathcal{A}]}{\rho_{\text{reg}}^{\frac{d-1}{d}}} + \frac{3}{4} \varepsilon^2$$

for some universal constant $C''_{\text{tilt}} \in (1, \infty)$. Hence, we may infer (252) from (256) after suitably selecting $M \geq M_0 \gg_{\mathcal{A}, C_\Lambda} 1$.

Conclusion: Since $\varepsilon \leq \varepsilon_{\text{reg}}$ by (212), the tilt excess estimates required by (220), (203) and (207) as well as the mass ratio estimates required by (203) and (207) follow from (235)–(236), (239)–(240) and (251)–(252), respectively. This eventually concludes the proof. \square

Proof of Lemma 22. The proof is split into two steps.

Step 1: We first claim that

$$(257) \quad \text{supp } \mu \cap \partial \Omega = \emptyset.$$

(Note that the weaker property $|\mu|_{\mathbb{S}^{d-1}}(\partial \Omega) = 0$ is immediate from Lemma 21 and the remark at the end of the proof of Theorem 20).

For a proof of (257), we argue by contradiction and assume that there exists $x_0 \in \text{supp } \mu \cap \partial\Omega$. Thanks to Lemma 21, the conclusions of Theorem 20 item ii) apply, i.e., $\text{supp } \mu \cap B_{\gamma_{reg}\rho_{reg}}(x_0)$ admits a graph representation with associated height function u in the precise sense of (209)–(211). Due to the first estimate of (211), we infer that for $\alpha_{reg} \in (0, \frac{\pi}{2})$ defined by $C_{reg}\varepsilon_{reg}^{\frac{1}{2(d-1)+3}} =: \tan(\alpha_{reg})$ it holds $P_{x_0 + \text{Tan}_{x_0}^\perp \partial\Omega}(\text{supp } \mu \cap B_{\gamma_{reg}\rho_{reg}}(x_0)) \supset ((x_0 + \text{Tan}_{x_0}^\perp \partial\Omega) \cap B_{\gamma_{reg}\rho_{reg} \cos(\alpha_{reg})}(x_0))$, so that in particular

$$(258) \quad |\mu|_{\mathbb{S}^{d-1}}(B_{\gamma_{reg}\rho_{reg}}(x_0)) \geq \omega_{d-1}(\gamma_{reg}\rho_{reg} \cos(\alpha_{reg}))^{d-1}.$$

By (16) and (8), we also know that $|\mu|_{\mathbb{S}^{d-1}}(\text{supp } \mu \cap B_{\gamma_{reg}\rho_{reg}}(x_0)) \leq E_{rel}[A, \mu|\mathcal{A}]$, so that assumption (197) and a suitable choice $M \gg_{\mathcal{A}, C_\Lambda} 1$ imply

$$(259) \quad |\mu|_{\mathbb{S}^{d-1}}(B_{\gamma_{reg}\rho_{reg}}(x_0)) \leq \frac{1}{2}\omega_{d-1}(\gamma_{reg}\rho_{reg} \cos(\alpha_{reg}))^{d-1},$$

thus contradicting (258).

Step 2: By compactness of $\text{supp } \mu$ and $\partial\Omega$ it now follows from the first step that $\text{dist}(\text{supp } \mu, \partial\Omega) > 0$. Hence, the second identity of (224) is immediate from (199), and based on that, one obtains (223) simply from (227) and [15, Definition 3, item (ii)]. Since the canonically associated general varifold $\widehat{\mu}$ of μ is $(d-1)$ -integer rectifiable, the first identity of (224) follows now in turn from (223) and $|\widehat{\mu}|_{\mathbf{G}(d, d-1)} = |\mu|_{\mathbb{S}^{d-1}}$.

It remains to verify the first inclusion of (222). To this end, one may exploit a contradiction argument being essentially analogous to the one conducted in Step 1 of this proof. Indeed, one works with the conclusions of Theorem 20 item i) instead of the ones from item ii), which are applicable due to the already established validity of (199) and Lemma 21, and exploits in addition property (214) for $\rho = \gamma_{reg}\rho_{reg}$ to ensure that $|\mu|_{\mathbb{S}^{d-1}}(B_{\gamma_{reg}\rho_{reg}}(x_0)) \leq 4E_{rel}[\mu, A|\mathcal{A}]$ for any $x_0 \in \{|\xi| \leq 1/2\}$. \square

Proof of Lemma 23. Denote by $(\mathcal{J}_k)_{k=1, \dots, K}$ the finitely many connected components of $\partial\mathcal{A}$. We now argue in four steps.

Step 1: We claim that for each $k \in \{1, \dots, K\}$ it holds

$$(260) \quad \mathcal{H}^{d-1}(\partial^* A \cap B_\ell(\mathcal{J}_k)) > 0.$$

If this is not the case, fix $\bar{k} \in \{1, \dots, K\}$ with $\mathcal{H}^{d-1}(\partial^* A \cap B_\ell(\mathcal{J}_{\bar{k}})) = 0$. Let $\mathcal{A}_{\bar{k}} \subset \mathcal{A}$ such that $\partial\mathcal{A}_{\bar{k}} = \mathcal{J}_{\bar{k}}$, so that by (222) and (16) it holds

$$\mathcal{H}^{d-1}(\partial^* A \cap \{\text{dist}(\cdot, \mathcal{A}_{\bar{k}}) < \ell\}) = 0.$$

Hence, by relative isoperimetric inequality $\text{vol}(A \cap \{\text{dist}(\cdot, \mathcal{A}_{\bar{k}}) < \ell\}) = 0$ which in turn implies

$$(261) \quad \|\chi_A - \chi_{\mathcal{A}}\|_{L^1}^2 \geq \text{vol}(\mathcal{A}_{\bar{k}})^2.$$

However, it also holds

$$(262) \quad \|\chi_A - \chi_{\mathcal{A}}\|_{L^1}^2 \leq C_{vol}\ell^{d+1}E_{vol}[A|\mathcal{A}]$$

for some universal constant $C_{vol} \in (1, \infty)$. Hence, assumption (197) together with the previous two displays guarantees $\min\{\text{vol}(\mathcal{A}_k)^2 : k = 1, \dots, K\} \leq \frac{C_{vol}}{M}\ell^{2d}$, so that choosing $M \gg_{\mathcal{A}} 1$ sufficiently large yields a contradiction.

Step 2: Fix $k \in \{1, \dots, K\}$. We claim that for all $y_0 \in \mathcal{J}_k$ there exists a local height function

$$(263) \quad h_{y_0}: \mathcal{J}_k \cap B_{\tilde{C}^{-1}\gamma_{reg}\rho_{reg}}(y_0) \rightarrow (-\ell, \ell) \text{ of class } C^{1,1-\frac{d-1}{4}} \cap H^2$$

such that

$$(264) \quad \left\{ y + h_{y_0}(y)n_{\partial\mathcal{A}}(y): y \in B_{\tilde{C}^{-1}\gamma_{reg}\rho_{reg}}(y_0) \cap \mathcal{J}_k \right\} \subset \partial^* A,$$

where \tilde{C} is the constant from (212). In particular, for all $y_0 \in \mathcal{J}_k$

$$(265) \quad \#\{s \in (-\ell, \ell): y_0 + sn_{\partial\mathcal{A}}(y_0) \in \partial^* A\} \geq 1.$$

For a proof of (263)–(264), we first choose $\tilde{x} \in \partial^* A \cap B_\ell(\mathcal{J}_k)$, where the latter set is indeed non-empty due to the first step. By (224) and Lemma 21, we may apply Theorem 20 item i) meaning there exists a $C^{1,1-\frac{d-1}{4}}$ function $u: (\tilde{x} + \text{Tan}_{\tilde{y}}\partial\mathcal{A}) \cap B_{\gamma_{reg}\rho_{reg}}(\tilde{x}) \rightarrow \mathbb{R}$, where $\tilde{y} := P_{\partial\mathcal{A}}(\tilde{x})$, such that $\partial^* A$ can be represented within $B_{\gamma_{reg}\rho_{reg}}(\tilde{x})$ by means of u in the sense of (205) and that, thanks to (206) and (220)–(221), it holds

$$(266) \quad \sup |\nabla_{\tilde{x} + \text{Tan}_{\tilde{y}}\partial\mathcal{A}} u| \leq 3C_{reg}\varepsilon =: \tan \alpha_{reg}.$$

In particular,

$$(267) \quad P_{\tilde{x} + \text{Tan}_{\tilde{y}}\partial\mathcal{A}}(\partial^* A \cap B_{\gamma_{reg}\rho_{reg}}(\tilde{x})) \supset ((\tilde{x} + \text{Tan}_{\tilde{y}}\partial\mathcal{A}) \cap B_{(\cos \alpha_{reg})\gamma_{reg}\rho_{reg}}(\tilde{x})).$$

In preparation for the proof of the regularity claim of (263), we first show that

$$(268) \quad u \in (C^{1,1-\frac{d-1}{4}} \cap H^2)((\tilde{x} + \text{Tan}_{\tilde{y}}\partial\mathcal{A}) \cap B_{\frac{1}{4}(\cos \alpha_{reg})\gamma_{reg}\rho_{reg}}(\tilde{x})),$$

where only the asserted second-order Sobolev regularity requires further attention. In spirit of Subsection 5.5, let us introduce a change of variables

$$(269) \quad \Psi^u: \underbrace{\overbrace{(\tilde{x} + \text{Tan}_{\tilde{y}}\partial\mathcal{A}) \cap B_{(\cos \alpha_{reg})\gamma_{reg}\rho_{reg}}(\tilde{x})}^{\text{T}_{\tilde{x}}}}_{=: \mathcal{U}_{\tilde{x}}} \rightarrow \partial^* A \cap B_{\gamma_{reg}\rho_{reg}}(\tilde{y})$$

$$x \mapsto x + u(x)n_{\partial\mathcal{A}}(\tilde{y}).$$

Defining $a: (-1, \infty) \rightarrow \mathbb{R}$, $r \mapsto \frac{1}{\sqrt{1+r}}$, we will derive the claim $u \in H^2(\frac{1}{4}\mathcal{U}_{\tilde{x}})$ from two things: (i) within $\frac{1}{2}\mathcal{U}_{\tilde{x}}$, u is a weak solution of

$$(270) \quad \nabla_{\text{T}_{\tilde{x}}} \cdot (a(|\nabla_{\text{T}_{\tilde{x}}} u|^2)\nabla_{\text{T}_{\tilde{x}}} u) = (\text{H} \cdot n_{\partial^* A}) \circ \Psi^u \in L^2\left(\frac{1}{2}\mathcal{U}_{\tilde{x}}\right),$$

and (ii) a standard difference quotient argument applied to (270), where the non-linearity in (270) turns out to represent no obstruction due to an absorption argument facilitated by the estimate (266) and the smallness of ε , see (212).

In order to recover (270), fix $\varsigma \in C_{cpt}^\infty(\text{T}_{\tilde{x}})$ with $\text{supp } \varsigma \subset \frac{1}{2}\mathcal{U}_{\tilde{x}}$, fix $\bar{v} \in C^\infty(\mathbb{R}; [0, 1])$ and $c_{\alpha_{reg}} \in (0, 1)$ with $\text{supp } \bar{v} \subset [-(1+c_{\alpha_{reg}}), (1+c_{\alpha_{reg}})]$ and $\bar{v} \equiv 1$ on $[-1, 1]$ such that the cutoff $(\varsigma \circ P_{\text{T}_{\tilde{x}}})v$, where $v(x) := \bar{v}\left(\frac{\text{dist}(x, \text{T}_{\tilde{x}})}{\frac{1}{2}(\sin \alpha_{reg})\gamma_{reg}\rho_{reg}}\right)$, is supported within the ball $B_{\gamma_{reg}\rho_{reg}}(\tilde{x})$. Denoting by $P_{\text{T}_{\tilde{x}}}$ the nearest-point projection onto $\text{T}_{\tilde{x}}$, the idea now is to test (224) with $B_\varsigma = (\varsigma \circ P_{\text{T}_{\tilde{x}}})v n_{\partial\mathcal{A}}(\tilde{y}) \in C_{cpt}^\infty(B_{\gamma_{reg}\rho_{reg}}(\tilde{x}))$ and subsequently represent integrals over $\partial^* A$ as integrals over $\mathcal{U}_{\tilde{x}}$.

As $d\mathcal{H}^{d-1}\llcorner(\partial^*A \cap B_{\gamma_{reg}\rho_{reg}}(\tilde{x})) = \frac{1}{(n_{\partial^*A} \circ \Psi^u) \cdot n_{\partial\mathcal{A}}(\tilde{y})} d\mathcal{H}^{d-1}\llcorner P_{T_{\tilde{x}}}(\partial^*A \cap B_{\gamma_{reg}\rho_{reg}}(\tilde{x}))$ and $H = (H \cdot n_{\partial^*A})n_{\partial^*A}$, the right hand side of (224) for the test function B_ζ is given by

$$(271) \quad - \int_{\partial^*A} H \cdot B_\zeta d\mathcal{H}^{d-1} = - \int_{\frac{1}{2}\mathcal{U}_{\tilde{x}}} \zeta(H \cdot n_{\partial^*A}) \circ \Psi^u d\mathcal{H}^{d-1}.$$

By the properties of the cut-off functions, we further compute on $\partial^*A \cap B_{\gamma_{reg}\rho_{reg}}(\tilde{x})$

$$(272) \quad \nabla B = n_{\partial\mathcal{A}}(\tilde{y}) \otimes (\nabla_{T_{\tilde{x}}}\zeta) \circ P_{T_{\tilde{x}}}.$$

Moreover, throughout $\mathcal{U}_{\tilde{x}}$ it holds

$$(273) \quad n_{\partial^*A} \circ \Psi^u = \frac{1}{\sqrt{1 + |\nabla_{T_{\tilde{x}}}u|^2}} (n_{\partial\mathcal{A}}(\tilde{y}) - \nabla_{T_{\tilde{x}}}u).$$

Due to the previous two displays and the already mentioned representation of the coarea factor, the left hand side of (224) for the test function B_ζ is given by

$$(274) \quad \int_{\partial^*A} (\text{Id} - n_{\partial^*A} \otimes n_{\partial^*A}) : \nabla B_\zeta d\mathcal{H}^{d-1} = \int_{\frac{1}{2}\mathcal{U}_{\tilde{x}}} a(|\nabla_{T_{\tilde{x}}}u|^2) \nabla_{T_{\tilde{x}}}u \cdot \nabla_{T_{\tilde{x}}}\zeta d\mathcal{H}^{d-1}.$$

In summary, (271) and (274) imply the claim (270). Performing now a difference quotient argument for the PDE (270) (see, e.g., [10, Section 4.3]), one observes that in this scheme the term for which the difference quotient operates on the coefficient $a(|\nabla_{T_{\tilde{x}}}u|^2)$ is a perturbative one with respect to the L^2 -norm of the difference quotient of $\nabla_{T_{\tilde{x}}}u$ thanks to (266) and the smallness of ε (the latter in the form of (212)). In other words, one obtains (268) by standard arguments.

Based on the properties (266)–(268) of the auxiliary local height function u , we now show that (263)–(264) holds for $\tilde{y} = P_{\partial\mathcal{A}}(\tilde{x})$. Consider to this end an arbitrary but fixed $y \in B_{\tilde{C}^{-1}\gamma_{reg}\rho_{reg}}(\tilde{y})$. Because of (267) as well as the flatness condition (216) applied for $x_0 = \tilde{x}$, $y_0 = \tilde{y}$ and $\rho = \gamma_{reg}\rho_{reg}$, we may infer that the slice $\{y + sn_{\partial\mathcal{A}}(y) : s \in (-\ell, \ell)\}$ intersects ∂^*A at least once in the region where the latter is locally represented by u . By (266), we also know that, for each point x on the graph induced by u , the cone with axis $\text{Tan}_{\tilde{y}}\partial\mathcal{A}$, opening angle $2\alpha_{reg}$ and apex sitting at x contains the whole graph induced by u . However, thanks to the flatness condition (217), this in turn simply means that the slice $\{y + sn_{\partial\mathcal{A}}(y) : s \in (-\ell, \ell)\}$ intersects ∂^*A exactly once in the region where the latter is locally represented by u , so that the local parametrization of ∂^*A in the sense of (264) follows.

Furthermore, we note that $h_{\tilde{y}}$ inherits the regularity of u , see (268), by means of the following argument: the map $\iota : (\tilde{x} + \text{Tan}_{\tilde{y}}\partial\mathcal{A}) \cap B_{(\cos\alpha_{reg})\gamma_{reg}\rho_{reg}}(\tilde{x}) \rightarrow \mathcal{I}_k$ defined by $x \mapsto P_{\partial\mathcal{A}}(x + u(x)n_{\partial\mathcal{A}}(\tilde{y}))$ induces a chart for \mathcal{I}_k , so that the claim follows from the identity $h(\iota(x)) = s_{\partial\mathcal{A}}(x + u(x)n_{\partial\mathcal{A}}(\tilde{y}))$, $x \in \mathcal{U}_{\tilde{x}}$, and smoothness of the signed distance function $s_{\partial\mathcal{A}}$. In summary, we proved the claim of Step 2 for $y_0 = \tilde{y}$.

Fix now $y \in \mathcal{I}_k \setminus \{\tilde{y}\}$. As \mathcal{I}_k is connected and compact, the Hopf–Rinow theorem ensures that we may find a geodesic $\gamma_{\tilde{y} \rightarrow y}$ connecting the two distinct points \tilde{y} , $y \in \mathcal{I}_k$. We then equipartition $\gamma_{\tilde{y} \rightarrow y}$ into N geodesic segments $\gamma_{\tilde{y}_{n-1} \rightarrow \tilde{y}_n}$, $n \in \{1, \dots, N\}$, where $\tilde{y}_0 := \tilde{y}$, $\tilde{y}_N := y$ and $N \in \mathbb{N}$ is sufficiently large such that $\tilde{y}_n \in B_{\tilde{C}^{-1}\gamma_{reg}\rho_{reg}}(\tilde{y}_{n-1}) \cap \mathcal{I}_k$ for all $n \in \{1, \dots, N\}$. Because of the latter condition, we may first transfer the conclusion of the previous argument from the

starting point $\tilde{y}_0 = \tilde{y}$ to the next point \tilde{y}_1 , and thus also, after finitely many iterations, to the endpoint $\tilde{y}_N = y$. This concludes the proof of (264).

Step 3: Fix again $k \in \{1, \dots, K\}$. We now show that for all $y_0 \in \mathcal{J}_k$ it holds

$$(275) \quad \#\{s \in (-\ell, \ell) : y_0 + s n_{\partial \mathcal{A}}(y_0) \in \partial^* A\} = 1.$$

We assume by contradiction that there exists $y_0 \in \mathcal{J}_k$ such that (275) fails, i.e., by (265), that

$$(276) \quad \#\{s \in (-\ell, \ell) : y_0 + s n_{\partial \mathcal{A}}(y_0) \in \partial^* A\} > 1.$$

Hence, we may choose $x_0 \in (\partial^* A \cap B_\ell(\mathcal{J}_k)) \setminus \{\tilde{x}_0\}$, where we also defined $\tilde{x}_0 := y_0 + h_{y_0}(y_0) n_{\partial \mathcal{A}}(y_0)$. Around both points x_0 and \tilde{x}_0 , the local graph property of $\partial^* A$ in the precise sense of Theorem 20 item i) holds true (with respect to the same tangent space). In particular, $\partial^* A \cap B_{\gamma_{reg} \rho_{reg}}(x_0)$ is a subset of the points within $\partial^* A \cap B_{\rho_{reg}}(x_0)$ where $\partial^* A$ is hit more than once by slicing in normal direction relative to $y_0 + \text{Tan}_{y_0} \partial \mathcal{A}$ (or more precisely, relying on the notation introduced right before (242), $\partial^* A \cap B_{\gamma_{reg} \rho_{reg}}(x_0) \subset S_{x_0}^{(1)}$). Hence, by a slicing argument,

$$(277) \quad \mathcal{H}^{d-1}(\partial^* A \cap B_{\gamma_{reg} \rho_{reg}}(x_0)) \leq C_{rel} E_{rel}[A, \mu | \mathcal{A}]$$

for some universal constant $C_{rel} \in (1, \infty)$. Based on a suitably large choice of $M \gg_{\mathcal{A}, C_\Lambda} 1$, one may now obtain a contradiction analogously to Step 1 of the proof of Lemma 22 (i.e., by the analogues of (258) and (259)).

Step 4: Fix again $k \in \{1, \dots, K\}$. By compactness of \mathcal{J}_k , we may select finitely many points $(\tilde{y}_n)_{n=1, \dots, N}$ along \mathcal{J}_k such that $\mathcal{J}_k \subset \bigcup_{n=1}^N B_{\tilde{C}^{-1} \gamma_{reg} \rho_{reg}}(\tilde{y}_n) \cap \mathcal{J}_k$. It then follows from the previous two steps that the local graph representations at $\tilde{y}_1, \dots, \tilde{y}_N$ from Step 2 can be stitched together in a consistent manner by a subordinate partition of unity. In other words, there exists $J_k \subset \partial^* A$ with the property that J_k can be obtained globally as a graph over \mathcal{J}_k in the sense of (45) with regularity (44).

We may now conclude because $\partial^* A = \bigcup_{k=1}^K J_k$ due to Step 3. \square

Proof of Lemma 24. We proceed in two steps.

Step 1: Estimate for $\sup |h|$. Assume that there exists $x_0 \in \partial^* A$ such that

$$(278) \quad |h(P_{\partial \mathcal{A}}(x_0))| > \frac{\ell}{16C}.$$

By (224) and Lemma 21, we may again apply Theorem 20 item i) to write $\text{supp } \mu = \partial^* A$ within $B_{\gamma_{reg} \rho_{reg}}(x_0)$ as a graph over $T_{x_0} := x_0 + \text{Tan}_{P_{\partial \mathcal{A}}(x_0)} \partial \mathcal{A}$ with height function u , cf. (205). Because of (206), (220)–(221), and (212)–(213)

$$(279) \quad \sup |u| \leq \rho_{reg} \varepsilon \leq \frac{1}{4} \frac{\ell}{16C},$$

$$(280) \quad \sup |\nabla_{T_{x_0}} u| \leq 2C_{reg} \varepsilon =: \tan \alpha, \quad \alpha \in (0, \pi/2).$$

Together with the previous two estimates, the flatness property (218) implies that

$$(281) \quad \|\chi_A - \chi_{\mathcal{A}}\|_{L^1} \geq \omega_{d-1} (\gamma_{reg} \rho_{reg} \cos \alpha)^{d-1} \frac{1}{2} \frac{\ell}{16C}.$$

In combination with the coercivity estimate (262) and assumption (197), we may therefore choose $M \gg_{\mathcal{A}, C_\Lambda} 1$ to reach a contradiction.

Step 2: Estimate for $\sup |\nabla_{\partial\mathcal{A}} h|$. Consider $y_0 \in \partial\mathcal{A}$ and define $x_0 := y_0 + h(y_0)\mathbf{n}_{\partial\mathcal{A}}(y_0) \in \partial^*A$. Denote by \mathbf{t}_1 and \mathbf{t}_2 a choice of orthonormal principal curvature directions at y_0 with corresponding principal curvatures κ_1 and κ_2 , and let $\tilde{\mathbf{t}}_1$ and $\tilde{\mathbf{t}}_2$ be the unique unit length tangent vectors of ∂^*A at x_0 such that $\mathbf{t}_i \cdot \tilde{\mathbf{t}}_i > 0$ and $\tilde{\mathbf{t}}_i \in \{\alpha\mathbf{t}_i + \beta\mathbf{n}_{\partial\mathcal{A}}(y_0) : \alpha, \beta \in \mathbb{R}\}$ for both $i = 1, 2$. Fix now $i \in \{1, 2\}$.

Locally around y_0 , we may choose a curve $\gamma_i : (-\delta, \delta) \rightarrow \partial\mathcal{A}$ such that $\gamma_i(0) = y_0$ and $(\gamma_i)'(0) = \eta_i$. This also induces a curve in ∂^*A by

$$(282) \quad \gamma_i^h : (-\delta, \delta) \rightarrow \partial^*A, \quad \theta \mapsto \gamma_i(\theta) + h(\gamma_i(\theta))\mathbf{n}_{\partial\mathcal{A}}(\gamma_i(\theta)).$$

A straightforward computation yields

$$(283) \quad (\gamma_i^h)'(0) = (1 - h(y_0)\kappa_i)\mathbf{t}_i + (\mathbf{t}_i \cdot \nabla_{\partial\mathcal{A}})h(y_0)\mathbf{n}_{\partial\mathcal{A}}(y_0)$$

and

$$(284) \quad \frac{(\gamma_i^h)'(0)}{|(\gamma_i^h)'(0)|} \cdot \mathbf{t}_i = \frac{1}{\sqrt{1 + \left(\frac{(\mathbf{t}_i \cdot \nabla_{\partial\mathcal{A}})h(y_0)}{1 - h(y_0)\kappa_i}\right)^2}}.$$

In particular,

$$(285) \quad \frac{(\gamma_i^h)'(0)}{|(\gamma_i^h)'(0)|} = \tilde{\mathbf{t}}_i.$$

Instead of using the height function h , we may also represent ∂^*A locally around x_0 by means of a height function u in the sense of (205). Defining $\gamma_{x_0, i}(\theta) := x_0 + \theta\mathbf{t}_i$, we then obtain a second induced curve in ∂^*A by

$$(286) \quad \gamma_i^u : (-\delta, \delta) \rightarrow \partial^*A, \quad \theta \mapsto \gamma_{x_0, i}(\theta) + u(\gamma_{x_0, i}(\theta))\mathbf{n}_{\partial\mathcal{A}}(y_0),$$

for which one may compute

$$(287) \quad (\gamma_i^u)'(0) = \mathbf{t}_i + (\mathbf{t}_i \cdot \nabla_{\mathbb{T}_{x_0}})u(x_0)\mathbf{n}_{\partial\mathcal{A}}(y_0)$$

and

$$(288) \quad \frac{(\gamma_i^u)'(0)}{|(\gamma_i^u)'(0)|} \cdot \mathbf{t}_i = \frac{1}{\sqrt{1 + ((\mathbf{t}_i \cdot \nabla_{\mathbb{T}_{x_0}})u(x_0))^2}},$$

so that again

$$(289) \quad \frac{(\gamma_i^u)'(0)}{|(\gamma_i^u)'(0)|} = \tilde{\mathbf{t}}_i.$$

In total, we obtain from these two different ways of representing $\tilde{\mathbf{t}}_i$ that

$$(290) \quad |(\mathbf{t}_i \cdot \nabla_{\partial\mathcal{A}})h(y_0)| = |1 - h(y_0)\kappa_i| |(\mathbf{t}_i \cdot \nabla_{\mathbb{T}_{x_0}})u(x_0)|.$$


By the previous step, $|h(y_0)| \leq \frac{1}{16}\ell$, so that the estimate $|\kappa_i| \leq \frac{1}{\ell}$ together with the previous identity, (280) and (212) entails

$$(291) \quad |(\mathbf{t}_i \cdot \nabla_{\partial\mathcal{A}})h(y_0)| \leq \frac{17}{16} |(\mathbf{t}_i \cdot \nabla_{\mathbb{T}_{x_0}})u(x_0)| \leq \frac{1}{16C}.$$

This concludes the proof. \square

Proof of Proposition 5. Immediate from Lemma 22, Lemma 23 and Lemma 24. \square

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INSTITUTE OF SCIENCE AND TECHNOLOGY AUSTRIA (IST AUSTRIA), AM CAMPUS 1, 3400 KLOSTERNEUBURG, AUSTRIA

Email address: julian.fischer@ist.ac.at

HAUSDORFF CENTER FOR MATHEMATICS, UNIVERSITÄT BONN, ENDENICHER ALLEE 62, 53115 BONN, GERMANY

Email address: sebastian.hensel@hcm.uni-bonn.de

FACULTY OF MATHEMATICS, UNIVERSITY OF REGENSBURG, UNIVERSITÄTSSTRASSE 31, 93053 REGENSBURG, GERMANY

Email address: tim.laux@ur.de

WESTFÄLISCHE WILHELMS-UNIVERSITÄT MÜNSTER, ORLÉANSRING 10, 48149 MÜNSTER, GERMANY

Email address: theresa.simon@uni-muenster.de